Ludwig-Maximilians-Universität München

Mathematical Gauge Theory II

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Contents

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1 Introduction

1.1 The Seiberg-Witten Equations

Seiberg-Witten theory is the study of solutions to a set of partial differential equations, called the *Seiberg-Witten* or *monopole* equations. They are formulated on a smooth, closed, oriented 4-manifold, and read:

$$D_A^+\Phi = 0\ ,$$
$$F_{\hat{A}}^+ = \sigma(\Phi,\Phi) + \omega\ ,$$

They depend on the choice of a Spin^c-structure $\mathfrak{s}$ on the manifold as well as two parameters:

(i) a Riemannian metric g on X, and

(ii) an imaginary-valued self-dual 2-form ω on X.

They are interpreted as equations for a pair (A,Φ), where

- Φ is a *positive (or left-handed) Weyl spinor*: A section of the (rank two) *spinor bundle* V_+ determined by $\mathfrak{s}$,
- $\hat{A}$ is a $U(1)$-connection on the line bundle $L = \Lambda^2 V_+$, induced by a Spin^c-connection A on V_+,
- D_A^+ is the *Dirac operator* on V_+ induced by A,
- $F_{\hat{A}}^+$ is the self-dual part of the curvature of $\hat{A}$, and
- $\sigma(\Phi,\Phi)$ is a self-dual 2-form defined by the spinor Φ.

The Seiberg-Witten (SW) equations are non-linear PDEs, since $D_A^+\Phi$ contains the quadratic term $A\Phi$ while $\sigma(\Phi,\Phi)$ is quadratic in Φ. To study the solution space of the SW equations for a given $(\mathfrak{s}, g, \omega)$ we use $\mathcal{C}_\mathfrak{s}$, the affine space of all configurations (A,Φ), and $\mathcal{Z}_\mathfrak{s} = \mathcal{Z}_\mathfrak{s}(g,\omega) \subset \mathcal{C}_\mathfrak{s}$, the space of all solutions to the SW equations.
The space of configurations is acted on by the *gauge group* $\mathcal{G} = C^\infty(X, S^1)$. This gives rise to the spaces $\mathcal{B}_\mathfrak{s} = \mathcal{C}_\mathfrak{s}/\mathcal{G}$ and $\mathcal{M}_\mathfrak{s} = \mathcal{M}_\mathfrak{s}(g,\omega) = \mathcal{Z}_\mathfrak{s}(g,\omega)/\mathcal{G}$. The latter is called the *Seiberg-Witten moduli space*. For a generic choice of (g,ω) this is a smooth, finite-dimensional, closed, oriented manifold. Note that $\mathcal{B}_\mathfrak{s}$ is infinite-dimensional and that $\mathcal{M}_\mathfrak{s} \subset \mathcal{B}_\mathfrak{s}$. The *Seiberg-Witten invariants* are integrals of certain differential forms over the moduli space.

1.2 Physical Motivation

The study of the Seiberg-Witten equations was initiated by physicists in the context of supersymmetric field theory. In physics, a field theory on a manifold X is typically determined by a *Lagrangian* $\mathcal{L}$, which depends on A and Φ. Path integrals are integrals over functions on $\mathcal{C}_\mathfrak{s}$, weighted by $\exp(\int_X \mathcal{L})$.

In physics, configurations connected by a gauge transformation are regarded as equivalent (a principle called *gauge symmetry*) and therefore the path integral can be reduced to an integral over the quotient $\mathcal{B}_\mathfrak{s}$ (which is still infinite-dimensional). In the special field theory originally considered by Seiberg and Witten, there is a further reduction: In the limit of small coupling, the integral over $\mathcal{B}_\mathfrak{s}$ localizes to an integral over the (finite-dimensional) moduli space. Hence, the integrals are rigorously defined; they correspond to the SW invariants which we will construct.

It turns out that these path integrals do not depend on the choice of (g,ω). However, the SW invariants *can* depend on the smooth structure of X^4. Indeed, it is possible to construct 4-manifolds X and Y that are homeomorphic but admit different SW invariants, hence are not (oriented) diffeomorphic. This gives rise to a phenomenon called *exotic 4-manifolds*. During this course, we will proceed as follows:

(i) define and understand the structures appearing in the SW equations,

(ii) study the properties of the SW moduli space,

(iii) prove that the SW invariants do not depend on choice of (g,ω), and

(iv) use these invariants to study smooth 4-manifolds.

2 Spin and Spin^c Structures

2.1 Clifford Modules

Definition 2.1 (Clifford Module)**.** Let H be a finite-dimensional, real vector space with a Euclidean scalar product g. A Clifford Module for H is a finite dimensional complex vector space V equipped with a Hermitian scalar product together with a linear map $\gamma : H \to \mathrm{End}(V)$ satisfying

- $\gamma(v)^\dagger = -\gamma(v)$,
- $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2g(v,w)\mathrm{Id}_V$.

The elements of V are called *spinors* and the action $H \times V \to V$, $(v, \phi) \mapsto \gamma(v)\phi$ is called *Clifford multiplication*.

A Clifford module corresponds to a representation of the Clifford algebra $\mathrm{Cl}(H, g)$ on V, i.e. a homomorphism $\mathrm{Cl}(H, g) \to \mathrm{End}(V)$. A Clifford module is called *irreducible* if there are no nontrivial submodules. We will use the following result without proof:

Proposition 2.2. *If* $\dim H = 2m$ *then there is a unique (up to isomorphism), irreducible Clifford module* (V, γ) *with* $\dim V = 2^m$*. If* $\dim H = n = 2m + 1$ *then there exist two irreducible Clifford modules* (V, γ)*, and* $(V, -\gamma)$*, with* $\dim V = 2^m$.

Example 2.3 (Standard Clifford Module for $\mathbb{R}^4$)**.** Consider $\mathbb{R}^4$ with the Euclidean metric. Then for $V = \mathbb{C}^4$, we define γ as follows. Choose an ONB $\{e_0, e_1, e_2, e_3\}$ for $\mathbb{R}^4$. Then we set

$$\gamma(e_j) = A_j := \begin{pmatrix} 0 & -B_j^\dagger \\ B_j & 0 \end{pmatrix}$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To check that $(\mathbb{C}^4, \gamma)$ this defines a Clifford module for $(\mathbb{R}^4, g_{\text{std.}})$, we have to check that

- $\gamma(e_j)^\dagger = -\gamma(e_j)$
- $\gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = -2\delta_{ij}\mathrm{Id}_{\mathbb{C}^4}$.

This was done in exercise 1.1. Clifford multiplication $\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$ extends to the exterior algebra, i.e. Clifford multiplication by multivectors: $\Lambda^*\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$. For $i_1 < i_2 < \ldots < i_l$, we define

$$\gamma(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}) = \gamma(e_{i_1})\gamma(e_{i_2}) \cdots \gamma(e_{i_l}) .$$

Note that this definition does not work if two of the i_j's are the same, since $\gamma(e_i \wedge e_i) = \gamma(0) = 0$, but $\gamma(e_i)\gamma(e_i) = -\mathrm{Id}_V$. This means that $\Lambda^*H \ncong \mathrm{Cl}(H, g)$ as algebras even though $\Lambda^*H \cong \mathrm{Cl}(H, g)$ as vector spaces.

Lemma 2.4. *For the standard Clifford module for* $\mathbb{R}^4$*, we have*

$$\gamma(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -\mathrm{Id}_2 & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix} .$$

Hence we can split $\mathbb{C}^4 = \mathbb{C}^2_+ \otimes \mathbb{C}^2_-$ where the labels are *opposite* to the eigenspace decomposition of $\gamma(\mathrm{vol})$, i.e. $\gamma(\mathrm{vol})|_{\mathbb{C}^2_\pm} = \mp\mathrm{Id}_2$. $\mathbb{C}^4$ is called the space of *Dirac spinors*; $\mathbb{C}^2_+$ (respectively $\mathbb{C}^2_-$) is the space of *positive* or *left handed* (respectively *negative* or *right handed*) *Weyl spinors*.

Clifford multiplication has a nice relation to the Hodge star operator: Consider $(\mathbb{R}^n, g_{\text{euclid}})$ with the standard volume form, i.e. $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$, for $\{\alpha_i\}$ the dual basis to $\{e_i\}$, an orthonormal basis.

Definition 2.5 (Hodge Duality on $\mathbb{R}^n$)**.** Hodge duality is a map $* : \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{n-k}(\mathbb{R}^n)^*$, defined by

$$*(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k}) = (-1)^\sigma (\alpha_{j_1} \wedge \alpha_{j_2} \wedge \ldots \wedge \alpha_{j_{n-k}})$$

Here, σ is the sign of the permutation that takes $(i_1\ i_2\ \ldots\ i_k\ j_1\ j_2\ \ldots\ j_{n-k})$ to $(1\ 2\ \ldots\ n)$.

Example 2.6 ($n = 4$, $k = 1$)**.**

$$\begin{aligned} *\alpha_1 &= \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ , \\ *\alpha_2 &= -\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \ , \\ *\alpha_3 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \ , \\ *\alpha_4 &= -\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ . \end{aligned}$$

Lemma 2.7. $*^2 = \mathrm{Id}$ *on* $\Lambda^2(\mathbb{R}^4)^*$.

Proof. This simple computation is left as an exercise to the reader. □

Since $*$ fixes $\Lambda^2(\mathbb{R}^4)^*$, two-forms on $\mathbb{R}^4$ can be decomposed into a self-dual and anti self-dual part as

$$\omega = \omega_+ + \omega_- = \frac{1}{2}(\omega + *\omega) + \frac{1}{2}(\omega - *\omega)$$

In other words, there is an (orthogonal) decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda_+(\mathbb{R}^4)^* \oplus \Lambda_-(\mathbb{R}^4)^*$. If $\{e_0, \ldots e_3\}$ is an oriented, orthonormal basis of $\mathbb{R}^4$, we have the following basis for $\Lambda^2_\pm(R^4)$:

$$\begin{gathered} e_0 \wedge e_1 \pm e_2 \wedge e_3 \\ e_0 \wedge e_2 \mp e_1 \wedge e_3 \\ e_0 \wedge e_3 \pm e_1 \wedge e_2 \end{gathered}$$

Hence, $\dim \Lambda^2_\pm(\mathbb{R}^4)^* = 3$, and $\dim \Lambda^2(\mathbb{R}^4)^* = 6$. We end this section with two important observations regarding the standard Clifford module γ_0 for $\mathbb{R}^4$.

Lemma 2.8. *Under γ_0, the self dual 2-forms*

$$\begin{gathered} e_0 \wedge e_1 + e_2 \wedge e_3 \\ e_0 \wedge e_2 - e_1 \wedge e_3 \\ e_0 \wedge e_3 + e_1 \wedge e_2 \end{gathered}$$

act non-trivially on $\mathbb{C}^2_+$ as $2B_1$, $2B_2$ and $2B_3$ respectively, and are zero on $\mathbb{C}^2_-$. An analogous result holds for the basis set of $\Lambda^2_-(\mathbb{R}^4)^*$.

Proof. See exercise 1.2. □

As a consequence, we have the following.

Lemma 2.9. *γ_0 induces isomorphisms*

$$\begin{aligned} (\Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\mathbb{R}^4)) \otimes \mathbb{C} &\cong \mathrm{Hom}(\mathbb{C}^2_+, \mathbb{C}^2_-) \otimes \mathrm{Hom}(\mathbb{C}^2_-, \mathbb{C}^2_+) \ , \\ \Lambda^2_\pm(\mathbb{R}^4) \otimes \mathbb{C} &\cong \mathrm{End}_0(\mathbb{C}^2_\pm) \ , \\ \Lambda^4(\mathbb{R}^4) \otimes \mathbb{C} &\cong \mathbb{C}\ \mathrm{Id}_{\mathbb{C}^2_\pm} \ , \end{aligned}$$

where $\mathrm{End}_0(\mathbb{C}^2_\pm)$ is the space of trace-free endomorphisms of $\mathbb{C}^2_\pm$.

Definition 2.10 (Clifford Module Isomorphism)**.** An isomorphism of Clifford modules (V, γ) and (V', γ') for H is a linear isometry[1] $f : V \to V'$ such that $f \circ \gamma(v) = \gamma'(v) \circ f$ for all $v \in H$.

Lemma 2.11 (Schur)**.** *Let (V, γ) be an irreducible Clifford module of H. Then every automorphism of V is of the form $f = \lambda \mathrm{Id}_V$ for a constant $\lambda \in S^1$.*

Proof. Since f is an isomorphism of Clifford modules it is an isometry, hence unitary. As such, it can be diagonalized; each eigenvalue has unit norm. The requirement on f is that it commute with γ means that γ preserves the eigenspaces of f:

$$f(\phi) = \lambda_i \phi \implies \gamma(f(\phi)) = f(\gamma(\phi)) = \lambda_i(\gamma(\phi)) \; .$$

Thus, each eigenspace is a submodule. Since γ is irreducible, V must be an eigenspace of f, i.e. $f = \lambda \mathrm{Id}$. λ has unit norm, hence lies in S^1. □

2.2 The Sheaf Cohomology Point of View

2.2.1 Čech Cohomology

Let X be a manifold, and S a *sheaf* over X: if G is a Lie Group, S_G assigns to an open subset $U \subset M$ the continuous functions $U \to G$. We will mostly work with the Abelian groups S^1, $\mathbb{R}$ and $\mathbb{Z}$. For now, let S be any sheaf and consider a locally finite open covering $\mathcal{U} = \{U_a\}_{a \in A}$ of X. We define the *p-cochain groups* as the formal products

$$\begin{aligned}
C^0(\mathcal{U}; S) &= \prod_a S(U_a) \; , \\
C^1(\mathcal{U}; S) &= \prod_{a \neq b} S(U_a \cap U_b) \; , \\
&\;\;\vdots \\
C^p(\mathcal{U}; S) &= \prod_{\substack{a_0, \ldots, a_p \\ \text{pairwise different}}} S(U_{a_0} \cap U_{a_1} \cap \ldots \cap U_{a_p}) \; .
\end{aligned}$$

We now define the *coboundary* operator $\delta : C^p(\mathcal{U}; S) \to C^{p+1}(\mathcal{U}; S)$, given by

$$(\delta\sigma)_{a_0 \ldots a_{p+1}} \coloneqq \prod_{j=0}^{p+1} \sigma^{(-1)^j}_{a_0 \ldots \hat{a}_j \ldots a_{p+1}} \Big|_{U_{a_0} \cap \ldots \cap U_{a_{p+1}}}$$

where the hat denotes omission. We write out the coboundary operator for low values of p: If $\sigma = \{\sigma_a\} \in C^0$, then $(\delta\sigma)_{ab} = \sigma_b \sigma_a^{-1}|_{U_a \cap U_b}$. For $\sigma = \{\sigma_{ab}\} \in C^1$ we find $(\delta\sigma)_{abc} = \sigma_{bc}\sigma_{ac}^{-1}\sigma_{ab}|_{U_a \cap U_b \cap U_c}$, and so on.

Definition 2.12 (Čech Cohomology)**.** We call a p-cochain $\sigma \in C^p$ a *p-cocycle* if $\delta\sigma = 0$. The set of p-cocycles is denoted by $Z^p(\mathcal{U}; S)$. We say $\sigma \in C^p$ is a *p-coboundary*, i.e. an element of $B^p(\mathcal{U}; S)$ if $\sigma = \delta\tau$ for some $\tau \in C^{p-1}$. The Čech cohomology groups $\check{H}^p(X, S)$ are defined as the direct limit of $\check{H}^p(\mathcal{U}; S) \coloneqq \frac{Z^p(\mathcal{U}; S)}{B^p(\mathcal{U}; S)}$, in the direct limit as the cover $\mathcal{U}$ gets finer.

Remark 2.13. On sufficiently nice spaces (e.g. manifolds), Čech cohomology of the sheaves of locally constant functions into $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Z}$ is isomorphic to singular cohomology with corresponding coefficients.

2.2.2 Spin^c Structures

Let X be a manifold and $H \to X$ a real, oriented vector bundle equipped with a Euclidean bundle metric.

[1] A map $f : V \to V'$ between vector spaces is called an isometry if $h_{V'}(f(a), f(b)) = h_V(a, b)$ for every $a, b \in V$.

Definition 2.14 (Spin^c Structure)**.** A Spin^c-structure $\mathfrak{s}$ on $H \to X$ is a pair (V, γ) where $V \to X$ is a complex vector bundle with a Hermitian bundle metric, and γ a bundle homomorphism $\gamma : H \to \mathrm{End}(V)$ which is fiberwise a standard (irreducible) Clifford module.

Using Čech cohomology, we will find that the obstruction to the existence of Spin^c structures for $H \to X$ is a class $o(H) \in \check{H}^3(X;\mathbb{Z}) = H^3(X;\mathbb{Z})$. A Spin^c structure for H is constructed by by "gluing together" Spin^c-structures on the trivial bundles $H|_{U_a}$ (for a locally finite open covering $\{U_a\}_{a\in A}$ of X). More precisely, if we assume H has even rank then proposition 2.2 tells us that each $\mathfrak{s}_a$ is unique up to isomorphism. For each $U_{ab} = U_a \cap U_b$, choose an isomorphism $\phi_{ab} : \mathfrak{s}_a|_{U_{ab}} \to \mathfrak{s}_b|_{U_{ab}}$. Of course we can assume that $\phi_{aa} = \mathrm{Id}$ and $\phi_{ab} = \phi_{ba}^{-1}$.

This gluing procedure is (globally) consistent precisely if for every $U_{abc} = U_a \cap U_b \cap U_c$ we have $\phi_{ab}\phi_{bc} = \phi_{ac}$. Hence, the 2-cochain $\Psi_{abc} := \phi_{ab}\phi_{bc}\phi_{ca}$ is the identity for all $a,\ b,\ c$, if and only if the $\{\mathfrak{s}_a\}$ glue together to a Spin^c-structure for H. Since Ψ_{abc} is a fiberwise automorphism of $\mathfrak{s}_a|_{U_{abc}}$ we can apply Schur's lemma to view it as a collection of maps $\Psi_{abc} : U_{abc} \to S^1$ for all a, b, c pairwise disjoint. Hence, Ψ_{***} is a 2-cochain of S_{S^1}.

Lemma 2.15. $\{\Psi_{***}\}$ *is a 2-cocycle.*

Proof. This is just a computation:

$$(\delta\Psi)_{abcd} = \Psi_{bcd}\Psi_{acd}^{-1}\Psi_{abd}\Psi_{abc}^{-1} = \phi_{bc}\phi_{cd}\phi_{db}\phi_{ad}\phi_{dc}\phi_{ca}\phi_{ab}\phi_{bd}\phi_{da}\phi_{ac}\phi_{cb}\phi_{ba} = \mathrm{Id}|_{U_{abcd}}$$

because all the ϕ's cancel pairwise. □

We will not go into the technicalities, but simply claim:

Theorem 2.16. *The cohomology class $o(H) := [\Psi_{***}] \in \check{H}^2(X; S_{S^1})$ is well-defined and independent of the choices of isomorphisms ϕ_{**} and covering $\mathcal{U}$.*

Proof. We will not prove independence of covering—this is a standard technical thing that always needs to be taken care of when working with sheaf cohomology. For independence of choices of isomorphisms, consider different isomorphisms $\phi'_{ab} = \pi_{ab}\phi_{ab}$ for $\pi_{ab} : U_{ab} \to S^1$. Then $\Phi'_{abc} = \pi_{ab}\pi_{bc}\pi_{ca}\Psi_{abc} = (\delta\pi)_{abc}\Psi_{abc}$ hence Ψ is modified only by a coboundary. □

The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$, where the last map is the exponential map $t \mapsto \exp(2\pi i t)$, induces a long exact sequence on the level of sheaf cohomology:

$$0 \longrightarrow \check{H}^0(X, S_{\mathbb{Z}}) \longrightarrow \dots \longrightarrow \check{H}^p(X, S_{\mathbb{R}}) \longrightarrow \check{H}^p(X, S_{S^1}) \longrightarrow \check{H}^{p+1}(X, S_{\mathbb{Z}}) \longrightarrow \dots$$

Now, using the fact that the higher (meaning $p > 0$) sheaf cohomology groups of $S_{\mathbb{R}}$ vanish ($S_{\mathbb{R}}$ is a *fine* sheaf), we see by exactness that $\check{H}^p(X, S_{S^1}) \cong \check{H}^{p+1}(X, S_{\mathbb{Z}})$ for $p > 0$. Observing that continuous maps into $\mathbb{Z}$ are in fact locally constant, we have $\check{H}^p(X; S_{\mathbb{Z}}) = H^p(X;\mathbb{Z})$ and therefore may view $o(H)$ as a class in $H^3(X;\mathbb{Z})$.

Corollary 2.17. *A* Spin^c*-structure for $H \to X$ exists if and only if $o(H) \in H^3(X;\mathbb{Z})$ is trivial.*

We turn to the question of uniqueness. For $\mathfrak{s}$, $\mathfrak{s}'$ Spin^c-structures for H, we may assume (after potentially passing to a common refined covering) that we have a covering $\{U_a\}$ such that $H|_{U_a}$is trivial and we have Spin^c structures $\mathfrak{s}|_{U_a}$ and $\mathfrak{s}'|_{U_a}$. From the isomorphisms $\tau_a : \mathfrak{s}_a \to \mathfrak{s}'_a$, we construct automorphisms $\sigma_{ab} = \tau_a^{-1}\tau_b : U_{ab} \to S^1$.

Lemma 2.18.

(i) $\{\sigma_{**}\}$ *is a 1-cocycle.*

(ii) *The cohomology class* $\delta(\mathfrak{s}, \mathfrak{s}') := [\sigma_{**}] \in \check{H}^1(X, S_{S^1})$ *well defined.*

Proof.

(i) This rolls straight out of the definition.

(ii) Again, we do not check independence of covering. If we have other isomorphisms $\tau'_a = \mu_a \tau_a$ for $\mu_a : U_a \to S^1$, we obtain a cochain $\{\mu_*\}$, hence σ_{ab} changes only by the coboundary $\delta\mu$.

□

It is precisely $\delta(\mathfrak{s}, \mathfrak{s}')$ which is the obstruction to finding an isomorphism $\mathfrak{s} \to \mathfrak{s}'$. We can say more:

Lemma 2.19. *Given a* Spin^c*-structure* $\mathfrak{s}$ *and a class* $\alpha \in \check{H}^1(X; S_{S^1})$*, there exists some* $\mathfrak{s}'$ *with* $\delta(\mathfrak{s}, \mathfrak{s}') = \alpha$.

This is proven as follows: It is a standard fact from the theory of sheaf cohomology that $\check{H}^1(X, S_G)$ is the set of isomorphism classes of G-principal bundles. In the case of S^1, the identification $S^1 = U(1)$ tells us that this is also the set of isomorphism classes of complex line bundles (the transition maps into $\mathbb{C}^*$ can be taken to map into $U(1)$ after picking a Hermitian metric).

Recall that $\check{H}^1(X; S_{S^1}) \cong H^2(X; \mathbb{Z})$ and that for a fixed complex line bundle L, the first Chern class (to be defined in section 3.3.1) $c_1(L)$ is a class in $H^2(X; \mathbb{Z})$. In fact, the map $\check{H}^1(X; S_{U(1)}) \to H^2(X; \mathbb{Z})$ is (up to sign) given by c_1: The first Chern class classifies complex line bundles up to isomorphism. Hence, $\alpha \in \check{H}^1(X; S_{S^1})$ corresponds to a line bundle L_α with $c_1(L_\alpha) = \alpha$. Hence, $V_{\mathfrak{s}'} = V_{\mathfrak{s}} \otimes L_\alpha$ is a natural candidate for a Spin^c structure such that $\delta(\mathfrak{s}, \mathfrak{s}') = \alpha$. The proof is now finished by the following lemma:

Lemma 2.20. *The pair* $(V_{\mathfrak{s}'}, \gamma_{\mathfrak{s}'})$ *has a Clifford module structure, where* $V_{\mathfrak{s}'} := V_{\mathfrak{s}} \otimes L_\delta$, $i : \mathrm{End}(V_{\mathfrak{s}}) \to \mathrm{End}(V_{\mathfrak{s}'})$ *is an isomorphism, and* $\gamma_{\mathfrak{s}'} = i \circ \gamma_{\mathfrak{s}}$.

Proof. Left as an exercise to the reader. □

The above discussion is summarized as follows:

Proposition 2.21. *Let* $H \to X$ *be a real, oriented vector bundle (i.e. associated to an* $SO(n)$*-principal bundle). Then*

(i) H *admits a* Spin^c *structure if and only if* $o(H) \in H^3(X; \mathbb{Z})$ *vanishes.*

(ii) *The set* $\mathrm{Spin}^c(H)$*, if non-empty, is an affine space over* $H^2(X; \mathbb{Z})$ *with the action of* $H^2(X; \mathbb{Z})$ *on* $\mathrm{Spin}^c(H)$ *given by* $V_{\mathfrak{s}} \mapsto V_{\mathfrak{s}} \otimes L_\alpha$ *for* $\alpha \in H^2(X; \mathbb{Z})$.

2.2.3 Spin Structures

Definition 2.22 (Conjugate Vector Space)**.** Let V be a complex vector space. Then the conjugate vector space $\bar{V}$ is defined as follows.

- As an Abelian group, $V = \bar{V}$.
- Scalar multiplication is given by $\mathbb{C} \times V \to V$, $(\lambda, v) \mapsto \bar{\lambda} v$.

Remark 2.23. Note that $\mathrm{Id} : V \to \bar{V}$ is a complex antilinear map and that $\mathrm{End}\, V = \mathrm{End}\, \bar{V}$, hence if V yields a Spin^c structure for some H, then so does $\mathrm{End}\, \bar{V}$.

If H is even-dimensional, this implies that $V \cong \bar{V}$ as Clifford modules:

Lemma 2.24. *If* H *is even-dimensional with* (V, γ) *the unique irreducible Clifford module, then there exists a* $\mathbb{C}$*-linear map* $J : V \to \bar{V}$ *(i.e. a* $\mathbb{C}$*-antilinear map* $V \to V$*) that commutes with* $\gamma(v)$ *for all* $v \in H$.

Definition 2.25 (Charge Conjugation)**.** If $V \to X$ is a complex vector bundle, then there exists a complex conjugate vector bundle $\bar{V} \to X$. Since $\operatorname{End} V \cong \operatorname{End} \bar{V}$, we define charge conjugation as the map

$$\begin{aligned} \operatorname{Spin}^c(H) &\longrightarrow \operatorname{Spin}^c(H) \\ (V,\gamma) = \mathfrak{s} &\longmapsto \bar{\mathfrak{s}} = (\bar{V},\gamma) \end{aligned}$$

By the discussion in the previous section, the following definition always makes sense:

Definition 2.26 (Characteristic Line Bundle)**.** A complex line bundle $L_\mathfrak{s}$ such that $\mathfrak{s} = \bar{\mathfrak{s}} \otimes L_\mathfrak{s}$ (unique up to isomorphism) is called the characteristic line bundle of $\mathfrak{s}$.

Clearly, $\mathfrak{s} \cong \bar{\mathfrak{s}}$ if and only if $L_\mathfrak{s}$ is trivial. In this case, we say that $\mathfrak{s}$ arises from a Spin structure:

Definition 2.27 (Spin Structure)**.** A Spin^c-structure $\mathfrak{s}$ together with an isomorphism $J : \bar{\mathfrak{s}} \to \mathfrak{s}$ is called a Spin structure.

The following is clear:

Lemma 2.28. *The* Spin^c*-structure* $\mathfrak{s}$ *induced by a* Spin *structure for* H *is unique up to isomorphism.* □

We will consider the existence of Spin structures in section 2.3.2. On the question of uniqueness, we only mention the following result.

Lemma 2.29. *Suppose* $(\mathfrak{s}, J)$ *and* $(\mathfrak{s}', J')$ *are spin structures for* $H \to X$*. Then the line bundle* L *satisfying* $\mathfrak{s} \cong \mathfrak{s}' \otimes L$ *is* 2*-torsion, i.e.* $2c_1(L) = 0 \in H^2(X;\mathbb{Z})$.

Remark 2.30. This comes from the fact that the space of Spin structures is an affine space over the space of real line bundles (which are 2-torsion, since they are equal to their own dual). Therefore, two Spin^c structures induced by Spin structures differ by a complexified real line bundle (in some sense, this can be traced back to the fact that S^1 is to Spin^c structures what $\mathbb{Z}_2$ is to Spin structures).

Since the universal coefficients theorem tells us that $\operatorname{Tor} H^2(X;\mathbb{Z}) \cong \operatorname{Tor} H_1(X;\mathbb{Z})$, we obtain a nice corollary.

Corollary 2.31. *If* $H_1(X;\mathbb{Z})$ *is torsion-free, e.g.* X^4 *is simply connected, two* Spin *structures* $(\mathfrak{s}, J)$ *and* $(\mathfrak{s}', J')$ *have to satisfy* $\mathfrak{s} \cong \mathfrak{s}'$.

Remark 2.32. Note that this is an isomorphism of the underlying Spin^c structures, not necessarily of Spin *structures*. In fact, T^4 is an example of a manifold with several Spin structures, but they all induce the same Spin^c structure. However, Spin structure are classified by $\check{H}^1(X;\mathbb{Z}_2)$ and therefore in case e.g. $\pi_1(X) = 1$, we have a(t most a) unique Spin structure.

2.3 The Principal Bundle Point of View

2.3.1 Spin^c Structures

In this section, we rephrase the existence of Spin^c structures using the formalism of principal bundles. We assume that n is even throughout. $\gamma_0 : \mathbb{R}^n \to \operatorname{End} \mathbb{C}^N$ denotes the standard Clifford module.

Definition 2.33 ($\operatorname{Spin}^c(n)$)**.** The Lie group $\operatorname{Spin}^c(n)$ is defined as the set of pairs $(\tau, \sigma) \in \mathrm{SO}(n) \times \mathrm{U}(N)$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\tau"] \arrow[d, "\gamma_0"'] & \mathbb{R}^n \arrow[d, "\gamma_0"] \\ \operatorname{End}\mathbb{C}^N \arrow[r, "\operatorname{Ad}\sigma"] & \operatorname{End}\mathbb{C}^N \end{tikzcd}$$

where $\operatorname{Ad}\sigma : \operatorname{End}\mathbb{C}^N \to \operatorname{End}\mathbb{C}^N$, $\mu \mapsto \sigma \circ \mu \circ \sigma^{-1}$ is the adjoint action.

Lemma 2.34. *The homomorphism* $\operatorname{Spin}^c(n) \to \operatorname{SO}(n)$, $(\tau, \sigma) \mapsto \tau$ *is surjective with kernel* $\{(\operatorname{Id}_n, S^1)\} \cong S^1$.

Proof. Let $\tau \in SO(n)$ be arbitrary. Then γ_0 and $\gamma_0 \circ \tau$ yield irreducible Clifford modules, hence are isomorphic (at least if n is even). Thus there must exist an isometry $\sigma \in U(n)$ such that $\sigma \circ \gamma_0(-) = \gamma_0 \circ \tau(-) \circ \sigma$. But then $\operatorname{Ad}\sigma \circ \gamma_0(-) = \gamma_0 \circ \tau(-)$, i.e. $(\tau, \sigma) \in \operatorname{Spin}^c(n)$.

For the kernel, assume (τ, σ) is mapped to $\operatorname{Id} \in \operatorname{SO}(n)$. Then clearly $\tau = \operatorname{Id}$ and we see that $\sigma \circ \gamma_0(-) \circ \sigma^{-1} = \gamma_0(-)$ hence σ commutes with γ_0, i.e. is an Clifford module isomorphism. Therefore it is given by an element $\lambda \in S^1$. (corresponding to a diagonal matrix with λ in the diagonal entries). ☐

Remark 2.35. One can show that $(\tau, \sigma) \in \operatorname{Spin}^c(n)$ if and only if $\sigma : \mathbb{C}^N \to \mathbb{C}^N$ is an isomorphism of Clifford modules covering the isometry $\tau : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 2.36. *Specifying a* Spin^c*-structure for an oriented Euclidean vector bundle* $H \to X$ *is equivalent to specifying a principal* $\operatorname{Spin}^c(n)$*-bundle* $Q \to X$ *together with an isomorphism* $Q/S^1 \cong \operatorname{Fr}(H)$*, where* $\operatorname{Fr}(H)$ *is the oriented, orthonormal frame bundle.*

Proof. Suppose we are given a Spin^c-structure $\mathfrak{s} = (V, \gamma)$, and want to define a $\operatorname{Spin}^c(n)$-bundle $Q \to X$. We consider the sets of pairs of orientation-preserving linear isometries

$$(t, s) \in \operatorname{Isom}(\mathbb{R}^n, H_x) \times \operatorname{Isom}(\mathbb{C}^N, V_x) \cong \operatorname{SO}(n) \times \operatorname{U}(N)$$

that make the following diagram commute

$$\begin{tikzcd} H_x \arrow[d, "\gamma_x"'] & \mathbb{R}^n \arrow[l, "t"'] \arrow[d, "\gamma_0"] \\ \operatorname{End}(V_x) & \operatorname{End}(\mathbb{C}^N) \arrow[l, "\operatorname{Ad}(s)"] \end{tikzcd}$$

and set them equal to the fibers Q_x over $x \in X$. Then each fiber is diffeomorphic to $\operatorname{Spin}^c(n)$ by construction, and $Q_x/S^1 \ni [t, s] \mapsto t$ is a fiberwise isomorphism to $\operatorname{Fr}(H)$ by lemma 2.34. For the other direction, given a $\operatorname{Spin}^c(n)$-bundle $Q \to X$, we know we have a representation of $\operatorname{Spin}^c(n)$ on $\mathbb{C}^N$ given by $(\tau, \sigma) \mapsto \sigma \in \operatorname{U}(N)$. This yields an associated vector bundle $V \to X$ with fiber $\mathbb{C}^N$. It remains to show that the standard Clifford module γ_0 induces a Clifford module γ for V; this is left as an exercise. ☐

Since a G-principal bundle $P \to X$ is defined by a set $\{\gamma_{**}\}$ of transition functions, the class $[\gamma_{**}] := [P] \in H^1(X; S_G)$ corresponds to the isomorphism class of the bundle[2] P. From lemma 2.34 we obtain the short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \operatorname{Spin}^c(n) \xrightarrow{p} \operatorname{SO}(n) \longrightarrow 1$$

which induces a short exact sequence on the level of sheaves and thereby a long exact sequence[3]:

$$\cdots \longrightarrow \check{H}^1(X; S_{\operatorname{Spin}^c(n)}) \xrightarrow{p} \check{H}^1(X; S_{\operatorname{SO}(n)}) \xrightarrow{\delta} \check{H}^2(X; S_{S^1}) \longrightarrow \cdots$$

[2]This works even if G in non-Abelian. In this case, $\check{H}^1(X; S_G)$ is only a set, not a group. It has a special base point corresponding to the isomorphism class of the trivial bundle $[X \times G]$.

[3]Here, we are sweeping some technicalities under the rug: The spaces should simply be interpreted as pointed sets of isomorphism classes of G-bundles.

where p is the projection $(\tau, \sigma) \mapsto \tau$ and δ is the connecting homomorphism. Suppose now that the $\mathrm{SO}(n)$-bundle $\mathrm{Fr}(H)$ has isomorphism class $[H] = [\gamma_{**}] \in \check{H}^1(X; S_{\mathrm{SO}(n)})$; the above long exact sequence tells us that a $\mathrm{Spin}^c(n)$-bundle Q exists if there a lift of $[\gamma_{**}]$ to an element in $\check{H}^1(X; S_{\mathrm{Spin}^c(n)})$.

On the other hand, $\{\phi_{**}\}$ defined in section 2.2.2 is a 1-cochain of $S_{\mathrm{Spin}^c(n)}$ that covers $[\gamma_{**}]$ under p. If $[\phi_{**}]$ is a cocycle, we get a $\mathrm{Spin}^c(n)$-bundle Q such that $p[Q] = [H] \in \check{H}^1(X; S_{\mathrm{SO}(n)})$. It is clear from the definition of $[\Psi_{***}] = o(H) \in \check{H}^2(X; S_{S^1})$ that $[\phi_{**}]$ is a cocycle if and only if $\delta[H] = o(H) = 0 \in H^2(X; S_{S^1})$, in accordance with 2.17. This just means that we have a Spin^c structure precisely if $[H]$ lifts not just to a cochain but a co*cycle*.

We can also recover the obstruction to uniqueness (lemma 2.18): Suppose that $[Q]$, $[Q']$ are lifts of H. Then $p[Q] = p[Q']$, hence $p([Q] - [Q']) = 0$, so $[Q] - [Q']$ is in the image of $j : \check{H}^1(X; S_{S^1}) \to \check{H}^1(X; S_{\mathrm{Spin}^c(n)})$. We then find some $\delta(\mathfrak{s}, \mathfrak{s}') \in \check{H}^1(X; S_{S^1})$ such that $j(\delta(\mathfrak{s}, \mathfrak{s}')) = Q - Q'$, as before.

2.3.2 Spin Structures and Stiefel-Whitney Classes

Consider $(\mathbb{C}^N, \gamma_0, J_0)$, the standard Clifford module for $\mathbb{R}^n$ with its charge conjugation map.

Definition 2.37 ($\mathrm{Spin}(n)$)**.** Recall that $\mathrm{Spin}^c(n)$ was defined as a certain set of pairs $(\tau, \sigma) \in \mathrm{SO}(n) \times \mathrm{U}(N)$. We define $\mathrm{Spin}(n)$ as the subgroup of $\mathrm{Spin}^c(n)$ of elements (τ, σ) such that σ commutes with J_0.

Lemma 2.38. *The homomorphism* $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, $(\tau, \sigma) \mapsto \tau$ *is surjective with kernel* $\{(\mathrm{Id}_n, \pm\mathrm{Id}_N)\} \cong \mathbb{Z}_2$.

Proof. Exercises 2.1–3. ☐

We therefore have the following short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1$$

Comparing this with

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1$$

we obtain the new description

$$\mathrm{Spin}^c(n) \cong (\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \,,$$

where $\mathbb{Z}_2$ identifies (τ, σ, λ) with $(\tau, -\sigma, -\lambda)$.

Lemma 2.39. *The map* $(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \to \mathrm{Spin}^c(n)$, $[\tau, \sigma, \lambda] \mapsto (\tau, \lambda\sigma)$ *is an isomorphism.*

Proof. Exercise 2.4. ☐

Analogous to Spin^c-structures, which we saw are lifts of an real, oriented vector bundle H to a $\mathrm{Spin}^c(n)$-bundle Q such that $Q/S^1 \cong \mathrm{Fr}\, H$, spin structures for H are lifts of $\mathrm{Fr}\, H$ to a $\mathrm{Spin}(n)$-bundle P such that $P/\mathbb{Z}_2 \cong \mathrm{Fr}(H)$. As before, we have a short exact sequence

$$0 \longrightarrow S_{\mathbb{Z}_2} \longrightarrow S_{\mathrm{Spin}(n)} \longrightarrow S_{\mathrm{SO}(n)} \longrightarrow 0$$

and hence a long exact sequence

$$\cdots \longrightarrow \check{H}^1(X; S_{\mathrm{Spin}(n)}) \xrightarrow{q} \check{H}^1(X; S_{\mathrm{SO}(n)}) \xrightarrow{\delta'} \check{H}^2(X; S_{\mathbb{Z}_2}) \cong H^2(X; \mathbb{Z}_2) \longrightarrow \cdots$$

and, arguing as before, we see that a lift $[P]$ for H exists if and only if $\delta'[H] = 0 \in H^2(X; \mathbb{Z}_2)$.

Definition 2.40 (Second Stiefel-Whitney Class)**.** Let H be a real, oriented vector bundle, and $\mathrm{Fr}\, H$ its $\mathrm{SO}(n)$-principal frame bundle. The second Stiefel-Whitney class $w_2(H)$ of H is defined as $w_2(H) = \delta'[H] \in H^2(X; \mathbb{Z}_2)$.

In terms of w_2, the above discussion can be summarized as:

Proposition 2.41. *A manifold is* Spin *if and only if it(s tangent bundle) has vanishing second Stiefel-Whitney class.*

Let us now examine the relation between $w_2(H)$ and the obstruction class for Spin^c-structures.

Lemma 2.42. *The obstruction class $o(H) \in \check{H}^2(X; S_{S^1})$ is the image of w_2 under $\iota^* : H^2(X;\mathbb{Z}_2) \to \check{H}^2(X; S_{S^1})$.*

Proof. The commutative ladder

$$\begin{tikzcd}
1 \arrow[r] & S^1 \arrow[r] & \mathrm{Spin}^c(n) \arrow[r] & \mathrm{SO}(n) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[u, hook, "\iota"] & \mathrm{Spin}(n) \arrow[r] \arrow[u, hook] & \mathrm{SO}(n) \arrow[r] \arrow[u, "\cong"] & 1
\end{tikzcd}$$

gives rise to

$$\begin{tikzcd}
\check{H}^1(X, S_{\mathrm{Spin}^c(n)}) \arrow[r] & \check{H}^1(X; S_{\mathrm{SO}(n)}) \arrow[r, "\delta"] & \check{H}^2(X; S_{S^1}) = H^3(X;\mathbb{Z}) \\
\check{H}^1(X; S_{\mathrm{Spin}(n)}) \arrow[r] \arrow[u] & \check{H}^1(X; S_{\mathrm{SO}(n)}) \arrow[r, "\delta'"] \arrow[u, equal] & H^2(X;\mathbb{Z}_2) \arrow[u, "\iota^*"]
\end{tikzcd}$$

Hence $\iota^*\delta'[H] = \delta[H] = o(H) = \iota^* w_2[H]$. ☐

Lemma 2.43. *The map $\iota : H^2(X;\mathbb{Z}_2) \to \check{H}^2(X; S_{S^1}) \cong H^3(X;\mathbb{Z})$ is equal to the* Bockstein homomorphism *$\beta : H^2(X;\mathbb{Z}_2) \to H^3(X;\mathbb{Z})$ induced by the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

Proof. Consider the commutative ladder

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"'] \arrow[u, equal] & \mathbb{Z} \arrow[r, "\text{mod } 2"'] \arrow[u, hook, "\cdot\frac{1}{2}"'] & \mathbb{Z}_2 \arrow[r] \arrow[u, hook, "\iota"] & 0
\end{tikzcd}$$

Note that square on the right commutes since $\exp(2\pi i k/2) = (-1)^k$. On the level of cohomology we obtain

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & H^2(X; S_{S^1}) \arrow[r, "\cong"] & H^3(X;\mathbb{Z}) \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & H^2(X;\mathbb{Z}) \arrow[r, "\text{mod } 2"] & H^2(X;\mathbb{Z}_2) \arrow[r, "\beta"] \arrow[u, "\iota^*"] & H^3(X;\mathbb{Z}) \arrow[r] \arrow[u, equal] & H^3(X;\mathbb{Z}) \arrow[r] & \dots
\end{tikzcd}$$

Under the identification $H^2(X; S_{S^1}) \cong H^3(X;\mathbb{Z})$, we see that $\iota^* = \beta$. ☐

The upshot of this analysis can be stated as follows.

Proposition 2.44. *The obstruction class $o(H) = \beta(w_2(H))$ vanishes if and only if $w_2(H)$ has a lift to $H^2(X;\mathbb{Z})$, that is, if $w_2(H)$ is in the image of $H^2(X;\mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(X;\mathbb{Z}_2)$.*

Corollary 2.45. *A Spin^c-structure for H exists if and only if $w_2(H)$ is the $\bmod 2$ reduction of a class in $H^2(X;\mathbb{Z})$.*

2.4 $\mathrm{Spin}(4)$ **and** $\mathrm{Spin}^c(4)$

Recall that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| \, a,\ b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}$$

As a real vector space, we can view $\mathbb{R}^4$ as

$$\mathbb{R}^4 \cong \left\{ \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \, \middle| \, u,\ v \in \mathbb{C} \right\} \cong \mathbb{C}^2$$

and the action $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R} \to \mathbb{R}$, $(h_-, h_+, x) \mapsto h_- x h_+^{-1}$ defines a surjective homomorphism ϕ : $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ with kernel $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\} \cong \mathbb{Z}_2$. Hence,

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) = \left\{ \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} \middle| \, B_+,\ B_- \in SU(2) \right\}$$

We remark that $\phi : \mathrm{Spin}(4) \to \mathrm{SO}(4)$ is the universal covering. To identify $\mathrm{Spin}^c(n)$, we recall $\mathrm{Spin}^c(n) \cong (\mathrm{Spin}(n) \times \mathrm{U}(1))/\mathbb{Z}_2$ and find

$$\begin{aligned} \mathrm{Spin}^c(4) &= (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/((1,1,1) \sim (-1,-1,-1)) \\ &= \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \middle| \, A_+, A_- \in \mathrm{SU}(2);\ \lambda \in \mathrm{U}(1) \right\} \end{aligned}$$

Let $H \to X$ be a real rank-four Euclidean vector bundle equipped with a Spin^c-structure, which we regard as a Spin^c-bundle Q. Then we can construct the associated Weyl spinor bundles $S_\pm$, where the positive and negative spinors correspond (fiberwise) to $\mathbb{C}^4 = \mathbb{C}^2_+ \oplus \mathbb{C}^2_-$, as vector bundles associated to Q. The representations used to associate $S_\pm$ to Q are

$$\begin{aligned} \rho_\pm : \mathrm{Spin}^c(4) &\longrightarrow \mathrm{U}(2) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\longmapsto \lambda A_\pm \end{aligned}$$

Finally, from lemma 2.39, we recall that the map $(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \to \mathrm{Spin}^c(n)$, $[\tau, \sigma, \lambda] \mapsto (\tau, \lambda\sigma)$ is an isomorphism. In particular, an element of $\mathrm{Spin}^c(n)$ unambiguously specifies a value of λ^2 and therefore the following representation is well-defined:

$$\begin{aligned} \chi : \mathrm{Spin}^c(4) &\longrightarrow \mathrm{U}(1) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\longmapsto \lambda^2 = \det(\lambda A_+) = \det(\lambda A_-) \end{aligned}$$

This representation associates the *determinant line bundle* $L \cong \det(S_+) = \Lambda^2(S_+) \cong \det(S_-) = \Lambda^2(S_-)$ to Q. The conjugate bundles are defined in the obvious way: $\bar{S} = \bar{S}_+ \oplus \bar{S}_-$, associated to $\mathfrak{s}$ via

$$\begin{aligned} \bar{\rho}_\pm : \mathrm{Spin}^c(4) &\to \mathrm{U}(2) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\mapsto \overline{\lambda A}_\pm\ . \end{aligned}$$

Lemma 2.46. *There exists a fixed matrix $M \in \mathrm{SU}(2)$ such that $MAM^\dagger = \bar{A}$ for all $A \in \mathrm{SU}(2)$.*

Proof. The Pauli matrices $\{\sigma_i\}$ form a basis of $\mathrm{SU}(2)$, hence one can work out the conditions imposed by the equations $M\sigma_i M^\dagger = \bar{\sigma}_i$. We obtain

$$M = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$$

where we can pick either sign. $\square$

This implies that the fundamental representation of $\mathrm{SU}(2)$ and its complex conjugate are isomorphic. Moreover,

$$\lambda^2 \overline{\lambda A}_\pm = \lambda|\lambda|^2 \bar{A}_\pm = \lambda \bar{A}_\pm = M(\bar{\lambda} A_\pm)M^\dagger$$

so the following representations are also isomorphic: $\rho_\pm \cong \chi \otimes \bar{\rho}_\pm$. Hence have $S_\pm \cong \bar{S}_\pm \otimes L_{\mathfrak{s}}$ and therefore $L_{\mathfrak{s}}$ is the characteristic line bundle.

If the spin structure changes from $\mathfrak{s}$ to $\mathfrak{s}'$, the determinant bundles are related in the following way.

Lemma 2.47. *Suppose $\mathfrak{s}' = \mathfrak{s} \otimes E$ for a complex line bundle E. Then $L_{\mathfrak{s}'} = L_{\mathfrak{s}} \otimes E^2$. Hence, $c_1(L_{\mathfrak{s}'}) = c_1(L_{\mathfrak{s}}) + 2c_1(E)$.*

Proof. We have that $\mathfrak{s} = \bar{\mathfrak{s}} \otimes L_{\mathfrak{s}}$, and $\mathfrak{s}' = \bar{\mathfrak{s}'} \otimes L_{\mathfrak{s}'}$. Then since $\mathfrak{s}' = \mathfrak{s} \otimes E$, we have that

$$\mathfrak{s} \otimes E = \bar{\mathfrak{s}} \otimes \bar{E} \otimes L_{\mathfrak{s}'} = \bar{\mathfrak{s}} \otimes L_{\mathfrak{s}} \otimes E \ .$$

This in turn implies that

$$L_{\mathfrak{s}} \otimes E = L_{\mathfrak{s}'} \otimes \bar{E} \implies L_{\mathfrak{s}} \otimes E^2 = L_{\mathfrak{s}'} \otimes \bar{E} \otimes E \ .$$

Now notice that $\bar{E} \otimes E = E^* \otimes E = \mathrm{End}(E)$ (the first equality follows because E is a line bundle); the latter bundle is in fact trivial for every line bundle E (a global section is given by Id_E). Hence, $L_{\mathfrak{s}'} = L_{\mathfrak{s}} \otimes E^2$. $\square$

As a corollary, we see that $c_1(L_{\mathfrak{s}}) \mod 2 \in H^1(X;\mathbb{Z}_2)$ does not depend on the choice of $\mathfrak{s}$. We also note the following, without proof:

Proposition 2.48. $c_1(L_{\mathfrak{s}}) \equiv w_2(H) \mod 2$.

2.5 Spin^c-Connections and Dirac Operators

Let $H \to X$ be a real, oriented vector bundle equipped with a Euclidean metric g and a metric-compatible connection ∇^B. Assume H admits a Spin^c-structure $V \to X$ with Hermitian metric h (we will indicate Clifford multiplication by a dot).

Definition 2.49 (Spin^c-connection)**.** A Spin^c-connection ∇^A on V is a covariant derivative which is

(i) Hermitian, i.e. $L_Y h(\Phi, \Psi) = h(\nabla^A_Y \Phi, \Psi) + h(\Phi, \nabla^A_Y \Psi)$ for every $Y \in \mathfrak{X}(M)$ and $\Phi, \Psi \in \Gamma(V)$.

(ii) Compatible with ∇^B and Clifford multiplication in the sense that for every $Y \in \mathfrak{X}(M), \Phi \in \Gamma(V)$ and $T \in \Gamma(M)$ we have

$$\nabla^A_Y(T \cdot \Phi) = (\nabla^B_Y T) \cdot \Phi + T \cdot (\nabla^A_Y \Phi)$$

In case the first term vanishes, we have an obvious simplification:

Lemma 2.50. *Let ∇^A be a Spin^c-connection and T a parallel section of H with respect to ∇^B along the flow of a vector field Y. Then $\nabla^A_Y(T \cdot \Phi) = T \cdot \nabla^A_Y \Phi$.* $\square$

This is sometimes useful when verifying identities pointwise (where one may choose a local frame of parallel sections).

2.5.1 The Dirac Operator on $\mathbb{R}^n$

Loosely speaking, the Dirac operator is the "square root" of the Laplace operator. Let us try to formalize this idea: for a function $\phi : \mathbb{R}^n \to \mathbb{C}^N$, the Dirac operator D can be written in a basis as

$$D\Phi = \sum_{i=1}^{n} A_i \frac{\partial \Phi}{\partial x_i}$$

where the A_i's are constant, complex $N \times N$ matrices. The Laplace operator on $\mathbb{R}^n$ is given by

$$\Delta\Phi = -\sum_{i=1}^{n} \mathrm{Id}_N \frac{\partial^2 \Phi}{\partial x_i^2}$$

Imposing $D^2 = \Delta$, we obtain

$$\begin{aligned} D^2\Phi &= \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} D\Phi = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} \sum_{i=1}^{n} A_i \frac{\partial \Phi}{\partial x_i} \\ &= \sum_{i,j=1}^{n} A_j A_i \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \overset{!}{=} -\sum_{i=1}^{n} \mathrm{Id}_N \frac{\partial^2 \Phi}{\partial x_i^2} \end{aligned}$$

which is equivalent to

$$A_i^2 = -\mathrm{Id}_N \qquad \text{and} \qquad A_j A_i + A_i A_j = 0 \quad \forall\, i \neq j \tag{2.1}$$

We also want D to be formally self-adjoint: For $\Phi,\ \Psi : \mathbb{R}^n \to \mathbb{C}^n$, we consider the L^2-scalar product

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \Phi^\dagger \Psi$$

Now, we require $\langle D\Phi, \Psi \rangle = \langle \Phi, D\Psi \rangle$:

Lemma 2.51. *D is formally self-adjoint if and only if $A_i^\dagger = -A_i$ (i.e. A_i is skew-adjoint).*

Proof. This is a simple computation.

$$\langle D\Phi, \Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \sum_i \frac{\partial \Phi^\dagger}{\partial x_i} A_i^\dagger \Psi$$

while on the other hand, integration by parts shows that

$$\langle \Phi, D\Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \Phi^\dagger \sum_i A_i \frac{\partial \Psi^\dagger}{\partial x_i} = -\int_{\mathbb{R}^n} \mathrm{d}^n x \sum_{i=1}^{n} \frac{\partial \Phi^\dagger}{\partial x_i} A_i \Psi$$

Hence, $\langle D\Phi, \Psi \rangle = \langle \Phi D\Psi \rangle$ if and only if $A_i^\dagger = -A_i$. □

Equation (2.1) together with the above lemma indicate that $A_i^\dagger A_i = \mathrm{Id}_N$ for each i, i.e. the A_i's are unitary.

2.5.2 The Dirac Operator on a Spinor Bundle

We specialize to the following set-up: Let $H = TX \to X$, with a Riemannian metric g. Let $\nabla^B = \nabla$, be the unique torsion-free[4], metric-compatible connection (called the Levi-Cività connection). Let $\mathfrak{s}$ be a Spin^c-structure on X.

[4] Recall that this condition means that for $Y, Z \in \mathfrak{X}(X)$, $\nabla_Y Z - \nabla_Z Y = [Y, Z]$.

Definition 2.52 (Dirac Operator)**.** If ∇^A is a Spin^c-connection on V, the Dirac operator $D_A : \Gamma(V) \to \Gamma(V)$ is defined as the composition

$$D_A : \Gamma(V) \xrightarrow{\nabla^A} \Gamma(T^*X \otimes V) \xrightarrow{g} \Gamma(TX \otimes V) \xrightarrow{\gamma_{\text{eval}}} \Gamma(V)$$

where γ_{eval} denotes γ composed with Clifford multiplication (evaluation).

Lemma 2.53. *Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of (T_pX, g_p). Then*

$$D_A\Phi = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\Phi$$

Remark 2.54. In physics, one typically writes $D\Phi = i\gamma^\mu\partial_\mu\Phi$. Clearly, γ^μ corresponds to Clifford multiplication by e_μ while ∂_μ corresponds to $\nabla^A_{e_\mu}$. The factor i arises from the different convention $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Proof of Lemma. Let $\{\omega_i\}$ be the dual basis to $\{e_i\}$. Then the covariant derivative in coordinates is given by

$$\nabla^A_Y\Phi = \sum_{i=1}^{n} Y_i\nabla^A_{e_i}\Phi = \sum_i \omega_i(Y)\nabla^A_{e_i}\Phi = \left(\sum_i \omega_i \otimes \nabla^A_{e_i}\Phi\right)(Y)$$

Hence, the Dirac operator is given by

$$\begin{aligned} D_A\Phi &= \gamma_{\text{eval}} \circ g \circ \nabla^A\Phi = \gamma_{\text{eval}} \circ g\left(\sum_{i=1}^{n} \omega_i \otimes \nabla^A_{e_i}\Phi\right) = \gamma_{\text{eval}}\left(\sum_{i=1}^{n} e_i \otimes \nabla^A_{e_i}\Phi\right) \\ &= \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\Phi \end{aligned}$$

□

Definition 2.55 (Hermitian Scalar Product of Spinors)**.** In the above setting, suppose (X, g) is closed. We define a Hermitian scalar product on the space of smooth sections $\Gamma(V)$ given by

$$\langle\Phi, \Psi\rangle = \int_X h(\Phi, \Psi)\mathrm{vol}_g$$

where h is the Hermitian scalar product on the spinor bundle.

Proposition 2.56. *With respect to the Hermitian scalar product, D_A is formally self-adjoint.*

The proof of this proposition relies on the following result:

Lemma 2.57. *For η the 1-form defined by $\eta(X) = h(X \cdot \Phi, \Psi)$, the following holds:*

$$h(D_A\Phi, \Psi) - h(\Phi, D_A\Psi) = * \,\mathrm{d} * \eta$$

Proof. We prove this pointwise. Let (e_j) be a local frame of V around $p \in X$ such that $(\nabla e_i)_p = 0$, i.e. the e_i are parallel in p. Using that ∇^A is a Spin^c connection, we have (in p):

$$\begin{aligned} h(D_A\Phi, \Psi) - h(\Phi, D_A\Psi) &= \sum_i \Big(h\big(\nabla_{e_i}(e_i \cdot \Phi), \Psi\big) - h\big(\Phi, e_i \cdot \nabla^A_{e_i}\Psi\big)\Big) \\ &= \sum_i \Big(h\big(\nabla_{e_i}(e_i \cdot \Phi), \Psi\big) + h\big(e_i \cdot \Phi, \nabla^A_{e_i}\Psi\big)\Big) \\ &= \sum_i L_{e_i}h(e_i \cdot \Phi, \Psi) = \sum_i L_{e_i}\eta(e_i) = * \,\mathrm{d} * \eta \end{aligned}$$

where we used that Clifford multiplication is skew-Hermitian to pass to the second line; the last step is justified by exercise 3.3. □

Proof of Proposition. This is now an easy application of Stokes' theorem:

$$\langle D_A\Phi, \Psi\rangle - \langle D_A\Phi, \Psi\rangle = \int_X * \,\mathrm{d} * \eta \,\mathrm{vol}_g = \int_X \mathrm{d} * \eta = 0$$

□

Example 2.58 (D_A on X^4)**.** Let us consider the case where X^4 is an oriented 4-manifold, $H = TX$ and $V = V_+ \oplus V_-$ the Clifford module defined by a Spin^c structure on X.

Lemma 2.59. *Every* Spin^c*-connection on* V *preserves* $V_\pm$*.*

Proof. The volume form is always parallel with respect to the Levi-Cività connection ∇. Recall that $V_\pm$ are the $\mp$-eigenspaces of V under Clifford multiplication by vol_g. Because vol_g is parallel, a Spin^c-connection must commute with this Clifford multiplication. Thus, we obtain

$$\mp\nabla^A_Y\Phi_\pm = \nabla^A_Y(\mathrm{vol}_g \cdot \Phi_\pm) = \mathrm{vol}_g \cdot \nabla^A_Y\Phi_\pm$$

hence $\nabla^A_Y\Phi_\pm \in \Gamma(V_\pm)$. □

Lemma 2.60. *Clifford multiplication with a tangent vector exchanges* V_+ *and* V_-

Proof. This follows from exercise 1.2, where we showed $(\Lambda^1\mathbb{R}^4 \oplus \Lambda^3\mathbb{R}^4) \otimes \mathbb{C} \cong \mathrm{End}(\mathbb{C}^2_+, \mathbb{C}^2_-) \oplus \mathrm{End}(\mathbb{C}^2_-, \mathbb{C}^2_+)$. □

Therefore, the Dirac operator decomposes as

$$D_A\Phi = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix} \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

where $D_A^\pm : \Gamma(V_\pm) \to \Gamma(V_\mp)$.

2.5.3 The Existence of Spin^c-Connections

In this section, we think of Spin^c-structures as $\mathrm{Spin}^c(n)$-principal bundles. Consider the $\mathrm{SO}(n)$-principal bundle $\mathrm{Fr}\, H$ and recall the bijective correspondence between connection 1-forms on a principal bundle and covariant derivatives on associated bundles. By this correspondence, the covariant derivative ∇^B defines a principal connection B on $\mathrm{Fr}(H)$, i.e. $B \in \Omega^1(\mathrm{Fr}(H), \mathfrak{so}(n)) = \Gamma(T^*\,\mathrm{Fr}(H) \otimes \mathfrak{so}(n))$ such that

(i) $r_g^* B = \mathrm{Ad}(g^{-1}) \circ B$ for every $g \in \mathrm{SO}(n)$, where r_g is right-multiplication by g

(ii) $B(\widetilde{W}) = W$ for all $W \in \mathfrak{so}(n)$, $\tilde{W}$ the fundamental vector field defined by $W \in \mathfrak{so}(n)$

Now let $A \in \Omega^1(Q, \mathfrak{spin}^c(n))$ be a connection on the $\mathrm{Spin}^c(n)$-bundle $Q \to X$; it defines a Hermitian covariant derivative ∇^A on $V \to X$, where $V \to X$ is the vector bundle associated to $Q \to X$ via the representation

$$\begin{aligned} \rho : \mathrm{Spin}^c(n) &\longrightarrow \mathrm{U}(N) \\ (\tau, \sigma) &\longmapsto \sigma \end{aligned}$$

Let $\varrho : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$, $(\tau, \sigma) \mapsto \tau$ be the surjective homomorphism defined in lemma 2.34. We denote the induced Lie algebra homomorphisms by ρ_* and ϱ_*.

We wish to answer the following question: In the formalism of principal bundles, what does is it mean for A that ∇^A is a Spin^c-connection? Which conditions are imposed on A? We have the following results.

Lemma 2.61. *For $Y \in \mathfrak{spin}^c(n)$,*

$$\varrho_*(Y)(t \cdot \phi) = \rho_*(Y)t \cdot \phi + t \cdot \varrho_*(Y)\phi$$

for all $t \in \mathbb{R}^n$, $\phi \in \mathbb{C}^N$.

Proof. By the definition of $\mathrm{Spin}^c(n)$, $g \in \mathrm{Spin}^c(n)$ satisfies the equation $\rho(g)(t \cdot \phi) = (\varrho(g)(t)) \cdot (\rho(g)\phi)$. Differentiating this, we obtain the result we are looking for. □

Proposition 2.62. *Let $\pi : Q \to \mathrm{Fr}\, H$ be the bundle map that identifies $Q/S^1 \cong \mathrm{Fr}\, H$. A Hermitian covariant derivative ∇^A is a Spin^c-connection if and only if the following diagram commutes:*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle D\pi}\downarrow & & \downarrow{\scriptstyle \varrho_*} \\ T\,\mathrm{Fr}(H) & \xrightarrow[B]{} & \mathfrak{so}(n) \end{array}$$

Proof. Let $s : U \to Q$ be a local section on an open set $U \subset X$. It induces a section $\pi \circ s$ of $\mathrm{Fr}\, H$. We have the associated bundles $V = Q \times_\rho \mathbb{C}^N$ and $H = \mathrm{Fr}\, H \times_{\text{std}} \mathbb{R}^n$. Let $\Phi = [s, \phi] \in \Gamma(V, U)$, $T = [t, \pi \circ s] \in \Gamma(H, U)$ and $Y \in \mathfrak{X}(X)$. We have the covariant derivatives

$$\nabla^A_Y \Phi = [s, L_Y\phi + \rho_*(s^*A(Y))\phi] \qquad \nabla^B_Y T = [\pi \circ s, L_Y t + ((\pi \circ s)^* B(Y))t]$$

omitting the $[s, \dots]$ and $[\pi \circ s, \dots]$ for simplicity, we have:

$$\begin{aligned} \nabla^A_Y(T \cdot \phi) &= L_Y(t \cdot \phi) + \rho_*(s^*A(Y))(t \cdot \phi) \\ (\nabla^B_Y T) \cdot \phi &= (L_Y t) \cdot \phi + ((\pi \circ s)^* B(Y)t) \cdot \phi \\ T \cdot \nabla^A_Y \phi &= t \cdot L_Y \phi + t \cdot \rho_*(s^*A(Y))\phi \end{aligned}$$

∇^A is a Spin^c-connection precisely if $\nabla^A_Y(T \cdot \Phi) + T \cdot \nabla^A_Y \Phi$. Setting $s^*A(Y) = a$, it is clear that all we need to show is

$$\rho_*(a)(t \cdot \phi) = ((\pi \circ s)^* B(Y)t) \cdot \phi + t \cdot \rho_*(a)\phi$$

Using the previous lemma, this is equivalent to requiring that $(\pi \circ s)^* B(Y) = \varrho_*(a)$. But this means exactly that $s^* \circ \pi^* B = B \circ D\pi \circ Ds = \varrho_* \circ A \circ Ds$, i.e. $B \circ D\pi = \varrho_* \circ A$, at least on the image of Ds, but that is all we need for it to be true for the covariant derivative. □

Corollary 2.63. *∇^A is a Spin^c-connection compatible with ∇^B if and only if ∇^B is associated to the principal connection A on Q via the representation $\varrho : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$. More explicitly, for $T = [\pi \circ s, t] \in \Gamma(H)$, where $s : U \subset X \to Q$ is a local section and $t : U \to \mathbb{R}^n$, we can write*

$$\nabla^B_Y T = [\pi \circ s, L_Y t + \varrho_*(s^*A(Y))t]$$

Let $L \coloneqq L_{\mathfrak{s}}$ denote the characteristic line bundle associated to Q via the homomorphism

$$\begin{aligned} \chi : \mathrm{Spin}^c(n) &\to \mathrm{U}(1) \\ [\tau, \sigma, \lambda] &\mapsto \lambda^2 \end{aligned}$$

Then $\chi_* : \mathfrak{spin}^c(n) \to \mathfrak{u}(1)$ is a Lie algebra homomorphism. We further define $\mathcal{L}$ as the $\mathrm{U}(1)$-principal bundle corresponding to L, i.e. $\mathcal{L}$ is the complex frame bundle $\mathrm{Fr}^{\mathbb{C}}(L)$ of L.

Proposition 2.64. *A connection A on Q induces a connection $\mathcal{A}$ on $\mathcal{L}$ such that the following diagram, where $c : Q \to \mathcal{L}$ is a bundle map, commutes.*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle Dc}\downarrow & & \downarrow{\scriptstyle \chi_*} \\ T\mathcal{L} & \xrightarrow[\mathcal{A}]{} & \mathfrak{u}(1) \end{array}$$

Since $\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times U(1)/\mathbb{Z}_2$, the Lie algebra is given by $\mathfrak{spin}^c(n) \cong \mathfrak{spin}(n) \oplus \mathfrak{u}(1) \cong \mathfrak{so}(n) \oplus \mathfrak{u}(1)$. The isomorphism $\mathfrak{spin}^c(n) \to \mathfrak{so}(n) \oplus \mathfrak{u}(1)$ is given by $(\rho \times \chi)_*$. This allows us to construct a connection on Q from connections on $\mathrm{Fr}\, H$ and $\mathcal{L}$, bringing us to the main result of this section:

Theorem 2.65. *Let B be a principal $\mathrm{SO}(n)$-connection on $\mathrm{Fr}(H)$, and $\mathcal{A}$ a principal $\mathrm{U}(1)$-connection on $\mathcal{L}$. Then there exists a unique principal $\mathrm{Spin}^c(n)$-connection A on Q such that the following diagrams commute.*

$$\begin{tikzcd} TQ \arrow[r, "A"] \arrow[d, "D\pi"'] & \mathfrak{spin}^c(n) \arrow[d, "\varrho_*"] \\ T\,\mathrm{Fr}(H) \arrow[r, "B"] & \mathfrak{so}(n) \end{tikzcd} \qquad \begin{tikzcd} TQ \arrow[r, "A"] \arrow[d, "Dc"'] & \mathfrak{spin}^c(n) \arrow[d, "\chi_*"] \\ T\mathcal{L} \arrow[r, "\mathcal{A}"] & \mathfrak{u}(1) \end{tikzcd}$$

Proof. Let us denote the $SO(n) \times U(1)$-bundle on X with fiber $\mathrm{Fr}\, H_x \times \mathcal{L}_x$ by $\mathrm{Fr}\, H \tilde{\times} \mathcal{L}$. Consider a connection $B \oplus \mathcal{A}$ on this bundle and the bundle map $\pi \tilde{\times} c : Q \to \mathrm{Fr}\, H \tilde{\times} \mathcal{L}$. This is in fact a 2:1 covering, but we only need it to be a local diffeomorphism. This is guaranteed by the fact that we have an isomorphism $(\rho \times \chi)_*$, which implies that $D(\pi \tilde{\times} c)$ is an isomorphism as a map between tangent spaces between each point, i.e. on the fibers of the tangent bundles. This allows us to define A to be the map that makes the following diagram commute:

$$\begin{tikzcd} TQ \arrow[r, "A"] \arrow[d, "\text{iso on fibers}"', "D(\pi \tilde{\times} c)"] & \mathfrak{spin}^c(n) \arrow[d, "\cong"', "(\rho \times \chi)_*"] \\ T(\mathrm{Fr}\, H \tilde{\times} \mathcal{L}) \arrow[r, "B \oplus \mathcal{A}"] & \mathfrak{so}(n) \oplus \mathfrak{u}(1) \end{tikzcd}$$

This uniquely defines A because the vertical maps are isomorphisms (on fibers). □

Corollary 2.66. *The choice of a metric connection ∇^B on H and a Hermitian connection $\nabla^{\mathcal{A}}$ on $\mathcal{L}$ determines a unique Spin^c-connection ∇^A on V.*

3 Some Background on Four-Manifolds

3.1 Classification Results in Dimensions One Through Three

Closed, connected, oriented, smooth (CCOS) manifolds form a very natural class of spaces and understanding and classifying them has been an important area of research for hundreds of years. Here, we give a very brief overview of some of the most important results for low-dimensional manifolds (that is, manifolds of dimension $n \leq 4$).

In one dimension, the only closed manifold is the circle: Every one-dimensional CCOS manifold is diffeomorphic to S^1. The two-dimensional case is already considerably more interesting: Let M be CCOS and of dimension two. Then $M \cong \Sigma_g$, for some unique $g \geq 0$, where g indicates the genus of the surface. The sphere S^2 has $g = 0$, the torus T^2 has genus $g = 1$ and for larger g, we have:

$$\Sigma_g = \underbrace{T^2\#\ldots\#T^2}_{\text{g copies}}$$

We can equivalently classify (oriented) surfaces by their first Betti number $b_1(M)$ or Euler characteristic χ via the relations $g = b_1(\Sigma_g)/2$ and $\chi(\Sigma_g) = 2 - 2g$.

For $n = 3$, things get much more difficult.

Definition 3.1 (Prime Manifold)**.** A manifold M is prime if $M \cong M_1\#M_2$ implies that one of the M_i is a sphere. Loosely speaking, this means that M has no non-trivial decomposition.

Prime manifolds derive their importance from the following theorem, which we will not prove:

Theorem 3.2 (Kneser, Milnor)**.** *Every M^3 has an essentially unique decomposition $M = M_1\#\ldots\#M_k$ with the property that each M_i is prime.*

Observe that in dimension two, the classification of CCOS manifolds has a nice connection to constant curvature metrics: There is a metric of constant (sectional) curvature $+1$ on S^2, 0 on T^2 and -1 on $\Sigma_{g\geq 2}$. Thurston envisioned something similar for 3-manifolds: Thurston's famous *geometrization conjecture* (now a theorem, due to Perelman) says that each prime M_i can be cast into "geometric" pieces, carrying one of the following eight geometries:

$$S^3,\ \mathbb{R}^3, H^3,\ \ S^2\times\mathbb{R},\ H^2\times\mathbb{R},\ \mathrm{Nil}^3,\ \mathrm{Sol}^3,\ \widetilde{\mathrm{SL}_2(\mathbb{R})}$$

A corollary is the following celebrated theorem:

Theorem 3.3 (Perelman)**.** *If M^3 is closed, connected and simply connected, $M^3 \cong S^3$.*

The proof establishes that M^3 must admit a metric of constant positive curvature and hence it must be a so-called *space form*, a quotient of S^3. Since $\pi_1(M^3) = 1$ by assumption, one concludes that $M^3 \cong S^3$.

One general theme in the understanding of 3-manifolds can be summarized in the following slogan or "meta-theorem": "*3-manifolds are controlled by the fundamental group*".

3.2 Dimension Four

3.2.1 The Intersection Form

It turns out that four dimensions is very different from three. We start by listing some basic examples of 4-dimensional manifolds

(i) S^4; $\mathbb{C}\mathrm{P}^2$; T^4 and other flat 4-manifolds, i.e. quotients of $\mathbb{R}^4$ by discrete, cocompact group actions: These are classified by π_1;

(ii) $M^3 \times S^1$, and more generally, S^1-bundles over M^3, and M^3 bundles over S^1;

(iii) $\Sigma_g \times \Sigma_h$, and more generally, Σ_h-bundles over Σ_g;

(iv) Iterated connected sums of all of the above.

These examples already show that the fundamental group does not constrain 4-manifolds much. It turns out that in dimension four, (co)homological invariants are more important. One of them is particularly powerful:

Definition 3.4 (Intersection Form)**.** Given M^4, CCOS, we define the intersection form as follows[5]

$$\begin{aligned} Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \langle \alpha \smile \beta, [M] \rangle \end{aligned}$$

If one works with $\mathbb{R}$ as coefficient ring, i.e. uses de Rham cohomology, the intersection form is given by $Q_M(\alpha, \beta) = \int_M \alpha \wedge \beta$. Q_M is bilinear and symmetric. By Poincaré duality[6], Q_M induces a non-degenerate symmetric bilinear form

$$Q_M : H^2(M;\mathbb{Z})/\operatorname{Tor} \times H^2(M;\mathbb{Z})/\operatorname{Tor} \to \mathbb{Z}$$

Remark 3.5.

(i) Using the universal coefficients theorem and Poincaré duality, it is not hard to show that the torsion in $H^2(M;\mathbb{Z})$ is the same as in $H_2(M;\mathbb{Z})$, $H_1(M;\mathbb{Z})$, and $H^3(M;\mathbb{Z})$. The other homology and cohomology groups ($H_0(M;\mathbb{Z})$, $H_3(M;\mathbb{Z})$, , $H_4(M;\mathbb{Z})$, $H^1(M;\mathbb{Z})$, and $H^4(M;\mathbb{Z})$) are all torsion-free. In particular, if M^4 is simply connected, there is no torsion at all in (co)homology. Therefore, the intersection form contains all the information about (co)homology in this case, since $H^1 \cong H_1 \cong H^3 \cong H_3$ is trivial.

(ii) The name *intersection form* comes from the fact that the Poincaré dual pairing $H_2(M;\mathbb{Z}) \otimes H_2(M;\mathbb{Z}) \to \mathbb{Z}$, $(a, b) \mapsto a \cdot b$ can be realized by counting intersection points of representing cycles. We will describe this in more detail soon.

Definition 3.6. A class $[c] \in H_2(M;\mathbb{Z})$ is *realized* by a surface Σ if there is a continuous map $f : \Sigma \to M$ such that $f_*[\Sigma] = [c]$.

Lemma 3.7. *If M is smooth and simply connected, then every class $[c] \in H^2(M;\mathbb{Z})$ is realized by a smooth immersed sphere, and also by a smoothly embedded surface (possibly of higher genus).*

Proof. If $\pi_1(M) = 1$, the Hurewicz theorem tells us that $\pi_2(M)$ surjects onto $H_2(M;\mathbb{Z})$. Hence, for every $[c] \in H_2(M;\mathbb{Z})$, there is a continuous $f : S^2 \to M$ such that $f_*[S^2] = [c]$. Since M is smooth, every homotopy class of maps from $S^2 \to M$ contains smooth maps so we may even take f smooth. We may homotope it to an immersion since $\dim S^2 < \dim M^4$.

If $f : \Sigma^2 \to M$ is a smooth immersion, then using transversality it can be taken to be an embedding away from a finite number of *transverse double points*. Near such a double point, the surface looks like $(\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$ in $\mathbb{C}^2$. We remove small balls (at the origins) from the copies of $\mathbb{C}$, and replace them by an annulus $S^1 \times [0, 1]$ that connects the boundary circles. This may change the genus, but resolves the intersection without altering the homology class (cf. exercise 4.2). Thus, we eventually obtain an embedded surface. □

Let Σ_1, $\Sigma_2 \subset M^4$ be smoothly embedded surfaces. We may assume (after homotoping "infinitesimally") that $\Sigma_1 \pitchfork \Sigma_2$, meaning that they only intersect transversely in a finite number of points. If $p \in \Sigma_1 \cap \Sigma_2$, then

[5] $[M]$ is the *fundamental class* associated to M, which corresponds to the positive generator of the homology group $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

[6] It states that if M is an n-dimensional oriented closed manifold, then the kth cohomology group of M is isomorphic to the $(n-k)$th homology group of M, for all integers k, i.e. $H^k(M) \cong H_{n-k}(M)$.

$T_p\Sigma_1 \oplus T_p\Sigma_2 = T_pM$ for dimensional reasons (combined with transversality); we assign ± 1 to p according to whether the orientations of Σ_1 and Σ_2 induce the given orientation on T_pM or not. Then we define

$$\Sigma_1 \cdot \Sigma_2 \coloneqq \sum_{p \in \Sigma_1 \cap \Sigma_2} \pm 1$$

This yields the Poincaré dual intersection pairing, which we will also denote by Q_M. Since Q_M is non-degenerate, symmetric, bilinear on $H^2(M;\mathbb{Z})/\operatorname{Tor}$, it is represented by a symmetric integer matrix with respect to a basis for second cohomology. Poincaré duality implies that it is unimodular, i.e. the representing matrix has determinant ± 1.

Example 3.8.

(i) Q_{S^4} is trivial because $H_2(S^4;\mathbb{Z}) = 0$.

(ii) $M = S^2 \times S^2$. Take $S^2 \times \{p\}$ and $\{p\} \times S^2$ as a basis for $H_2(S^2 \times S^2;\mathbb{Z})$. These spheres do not self-intersect transversely (given two copies of e.g. $S^2 \times \{p\}$, homotope one to $S^2 \times \{p\}$ to obtain an empty intersection). However, they intersect each other in $\{p\} \times \{p'\}$: The standard orientation on $S^2 \times S^2$ makes this intersection count as $+1$. Hence, we find:

$$Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: H$$

a so-called "hyperbolic pair".

(iii) $M = \mathbb{C}\mathrm{P}^2$. Then $H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}$ is generated by $\left[\mathbb{C}\mathrm{P}^1\right]$. With the standard orientation on $\mathbb{C}\mathrm{P}^2$, the intersection form $Q_{\mathbb{C}\mathrm{P}^2} = (1)$.

(iv) If M has a given orientation, then $\bar{M}$ denotes the same manifold with the opposite orientation. Its intersection form is given by $Q_{\bar{M}} = -Q_M$.

(v) An application of the Mayer-Vietoris sequence shows that $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2} = \begin{pmatrix} Q_{M_1} & 0 \\ 0 & Q_{M_2} \end{pmatrix}$.

(vi) From the three preceding statements, we see that $p\ \mathbb{C}\mathrm{P}^2 \#\ q\ \overline{CP^2} = \mathbb{C}\mathrm{P}^2 \# \dots \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2} \# \dots \overline{\mathbb{C}\mathrm{P}^2}$, the connected sum of p copies of $\mathbb{C}\mathrm{P}^2$ with q copies of $\overline{\mathbb{C}\mathrm{P}^2}$, has intersection form $p(1) \oplus q(-1) \coloneqq \operatorname{diag}(1, \dots, 1, -1, \dots, -1)$ with p times 1 and q times -1 on the diagonal.

3.2.2 Related Invariants and some Classification Results

The following properties of Q_M follow from basic linear algebra:

(i) $\operatorname{rank} Q_M = b_2(M)$.

(ii) Considering Q_M on $H^2(M;\mathbb{Z}) \otimes \mathbb{R} = H^2(M;\mathbb{R})$, we can diagonalize Q_M over $\mathbb{R}$ Indeed, over $\mathbb{R}$, Q_M is equal to some $p(1) \oplus q(-1)$ since symmetric, bilinear forms on $\mathbb{R}$-vector spaces are classified by their rank and signature.

Definition 3.9 (Signature of a 4-Manifold)**.** We define $\sigma(M) = \sigma(Q_M \otimes \mathbb{R}) = p - q$ as the signature of M.

It is conventional to define $b_2^+(M) = p$, $b_2^-(M) = q$. We list some basic properties of the signature:

(i) $\sigma\left(\bar{M}\right) = -\sigma(M)$.

(ii) $\sigma(M)$ is an oriented homotopy invariant (it is defined using only (co)homology and an orientation). Hence, if $\sigma(M) \neq 0$, there is no orientation-preserving homotopy equivalence $f : M \to \bar{M}$. In other words, M does not admit an orientation-reversing homotopy equivalence $f : M \to M$.

(iii) The Euler characteristic of M is given by

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{4} b_i(M) = b_0(M) - b_1(M) + b_2(M) - b_3(M) + b_4(M) \\ &= 2 - 2b_1(M) + b_2^+(M) + b_2^-(M) \\ &\equiv b_2^+(M) - b_2^-(M) = \sigma(M) \mod 2\end{aligned}$$

where we used the fact that M is connected and Poincaré duality in passing to the second line. Since $\chi(M) \equiv \sigma(M) \mod 2$, an odd Euler characteristic shows that M does not admit an orientation-reversing self-homotopy equivalence.

Example 3.10.

(i) Every intersection form $p(1) \oplus q(-1)$ is realized by $p\,\mathbb{C}\mathrm{P}^2 \# q\,\bar{\mathbb{C}\mathrm{P}}^2$.

(ii) $Q_{S^2\times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H \cong_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, $\sigma(S^2 \times S^2) = 0$. An explicit orientation-reversing self-diffeomorphism of $S^2 \times S^2$ is given by $(x, y) \mapsto (x, -y)$ (regarding $S^2 \subset \mathbb{R}^3$ as the unit sphere).

(iii) Since $\sigma(\mathbb{C}\mathrm{P}^2) = 1$, it admits no orientation-reversing self-homotopy equivalence.

Definition 3.11. We say that Q_M is *even* if $Q_M(\alpha, \alpha) \equiv 0 \mod 2$ for all $\alpha \in H^2(M;\mathbb{Z})$. Otherwise, we call Q_M *odd*.

Example 3.12.

(i) $Q_{S^2\times S^2} = H$ is even since $H(a\alpha_1 + b\alpha_2, a\alpha_1 + b\alpha_2) = 2abH(\alpha_1, \alpha_2) \equiv 0 \mod 2$, where the α_i's are the standard basis elements. This shows that $S^2 \times S^2$ (with either orientation) is not homotopy equivalent to $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$.

(ii) Recall the exceptional Lie algebra E_8 with Dynkin diagram corresponding to the following matrix:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

It is a unimodular, symmetric, even and negative-definite matrix. We will denote this matrix by E_8. Clearly, $\sigma(E_8) = -8$. It plays an important role in the following fundamental theorem, which we will not prove.

Theorem 3.13 (Hasse-Minkowski Classification)**.** *Every unimodular, indefinite, symmetric bilinear form is equivalent over $\mathbb{Z}$ to either $b_2^+(1) \oplus b_2^-(-1)$ (where $b_2^+, b_2^- \geq 1$) if the form is odd, or $aH \oplus bE_8$ if the form is even, where $b \in \mathbb{Z}$, $a \in \mathbb{Z}$ and $a \geq 1$ (since $H \cong_{\mathbb{Z}} -H$).*

In the latter case we have $b_2^+ = a$, $b_2^- = a + 8b$ if $b \geq 0$ or, if $b \leq 0$, $b_2^+ = a - 8b$, $b_2^- = a$. This shows:

Remark 3.14. If Q_M is indefinite and even, then $\sigma(M) = -8b \equiv 0 \mod 8$.

If Q_M is definite, the situation is considerably more complicated. However, for smooth manifolds there is a celebrated theorem, due to Donaldson, which we will prove later on using Seiberg-Witten theory:

Theorem 3.15 (Donaldson)**.** *If M is a CCOS 4-manifold with Q_M definite, then Q_M is diagonal over $\mathbb{Z}$.*

Corollary 3.16. *Because of unimodularity, Q_M has only $+1$'s or -1's on the diagonal in this case. In particular, even intersection forms do not occur in this setting.*

This tells us that, in the smooth category, definite intersection forms are extremely constrained. For indefinite forms, a major conjecture remains:

Conjecture 3.17 (The 11/8-Conjecture)**.** *If M is a CCOS, simply connected 4-manifold with Q_M even, then*

$$b_2(M) \geq \frac{11}{8}|\sigma(M)|$$

Remark 3.18. The Hasse-Minkowski classification shows that the only such intersection forms are $aH \oplus bE_8$ where $a \geq 1$ and $b \in \mathbb{Z}$. We obtain the equivalent condition

$$2a + 8|b| \geq \frac{11}{8}8|b| \iff 2a \geq 3|b|$$

Another way of stating this conjecture is that smooth structures on M do not exist if $b_2(M) < (11/8)|\sigma(M)|$. The weaker inequality $b_2(M) \geq 10/8|\sigma(M)|$ has already been proved (by Furuta, using Seiberg-Witten theory). The $11/8$ conjecture is sharp, at least for $a = 3$, as we will see when we discuss $K3$ surfaces.

Remark 3.19. There is a CCOS 4-manifold M with $\pi_1(M) = \mathbb{Z}_2$ and $Q_M = H \oplus E_8$, i.e. the condition that M is simply connected is necessary.

Example 3.20 (S^2-bundles over S^2)**.** One natural question to ask is whether there are non-trivial oriented S^2-bundles over S^2. Since a disk is contractible, any bundle over S^2 is the result of gluing two trivial bundles (one on each hemisphere) over a subset that deformation retracts onto the equator by a transition map $f : S^1 \to G$, where G is the structure group of the bundle. In our case $G = \mathrm{Diff}_+(S^2)$, i.e. the orientation-preserving diffeomorphisms of S^2. Since we are interested in maps f up to homotopy, we can use the fact that $\mathrm{Diff}_+(S^2) \simeq \mathrm{SO}(3)$ (proven by Smale). Therefore, $\pi_1(\mathrm{Diff}_+(S^2)) = \pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, because $\mathrm{SO}(3) \cong \mathbb{RP}^3$. We can thus draw the following conclusions.

- There are at most 2 different S^2-bundles over S^2 up to diffeomorphism.
- All S^2-bundles can be taken to have structure group $\mathrm{SO}(3)$. Every S^2-bundle is a unit sphere bundle in a rank 3 vector bundle.

Now let $E \to S^2$ be the oriented rank 2 vector bundle whose Euler class e (see 3.24) is the generator of $H^2(S^2; \mathbb{Z})$. Let $V = E \oplus \underline{\mathbb{R}}$. Choose a metric on E and define $M := S(V)$ the unit sphere bundle in the induced metric on V. The intersection $M \cap (S^2 \times \underline{\mathbb{R}})$ consists of two copies of S^2, embedded as sections of the S^2-bundle $M \to S^2$. On the other hand, the intersection $M \cap E$ is a circle bundle over S^2 with Euler class the generator of $H^2(S^2; \mathbb{Z})$. Since the Euler class classifies such bundle and this description matches the Hopf bundle $S^1 \to S^3 \to S^2$, we conclude that $M \cap E \cong S^3$, and the circle bundle $S^3 \to S^2$ is the *Hopf fibration*.

M is obtained by doubling the unit disc bundle in E, i.e. gluing two copies by their boundaries. As a double, M admits an orientation-reversing diffeomorphism and therefore $\sigma(M) = 0$. In fact, the disk bundles are precisely $\mathbb{CP}^2 \setminus B^4$—thinking of the latter space in terms of its standard cell decomposition, with the the inner half of the 4-dimensional cell removed (recall that the gluing map of the 4-cell is the Hopf fibration).

Therefore, $M \cong D(\mathbb{C}\mathrm{P}^2 \setminus B^4) = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$ and it has intersection form $Q_M = (1) \oplus (-1) \not\cong_{\mathbb{Z}} H$. In particular, it is not homotopy equivalent to $S^2 \times S^2$. This bundle represents the non-trivial S^2-bundle over S^2. Notice that the parity of the intersection form distinguishes between the this bundle and the trivial one.

We close this section with two fundamental results, both of which highlight the central role of the intersection form in 4-manifold theory.

Theorem 3.21 (Whitehead-Milnor)**.** *Two CCOS, simply connected 4-manifolds are homotopy equivalent if and only if their intersection forms are equivalent over $\mathbb{Z}$.*

Proof. If $M_1 \simeq M_2$, their intersection forms agree by homotopy invariance of (co)homoloy. The converse is more difficult. A first observation (proven using e.g. Morse theory) is that every CCOS, simply connected 4-manifold M^4 has the homotopy type of a finite CW complex. Let $M_0 = M \setminus e^4$ be the complement of a 4-cell. Then

$$H_k(M_0) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1, \geq 3 \\ \mathbb{Z}^m & k = 2 \end{cases}$$

where $m = b_2(M)$ since there is no torsion and the 3-skeleton of M and M_0 coincide. Because M is simply connected, $\pi_2(M)$ surjects onto $H_2(M_0)$ by Hurewicz' theorem. Pick a basis $\{e_j\}$ for $H_2(M_0)$ and continuous maps $f_j : S^2_j \to M_0$ such that $(f_j)_*[S^2_j] = e_j$. After homotoping some of the f_j's to ensure they all hit a fixed base point, they yield a map

$$f : \bigvee_{j=1}^{m} S^2 \to M_0$$

which induces an isomorphism on homology in every degree. It is a theorem due to Whitehead that such a map between simply connected CW complexes is a homotopy equivalence. We now have $M \simeq \bigvee_j S^2_j \cup_g e^4$, where $g : S^3 \to \bigvee_j S^2_j$ is the gluing map of the 4-cell. Therefore, the homotopy type of M is determined by $b_2(M)$ and the homotopy class of g. We start by considering some low values of m case-by-case:

(i) The case $m = 0$ is easy: g must be constant, hence $M \simeq S^4$.

(ii) If $m = 1$, we have a map $g : S^3 \to S^2$ hence $[g] \in \pi_3(S^2)$: To figure out $\pi_3(S^2)$. We will do this via the Thom-Pontryagin construction. First, we will try to find a nice representative of $[g]$. Think of S^2, S^3 as embedded in $\mathbb{R}^3, \mathbb{R}^4$. For every continuous map $g : S^3 \to S^2$, we can find a smooth map $\tilde{g} : S^3 \to \mathbb{R}^3$ such that $\|\tilde{g}(x) - g(x)\| < \epsilon$ for arbitrarily small $\epsilon > 0$. Making ϵ small enough, we see that $\tilde{g}$ avoids $0 \in \mathbb{R}^3$. By "pushing radially", we see that $\tilde{g}$ is smoothly homotopic to a map into S^2; call the resulting map G.

Now G can be homotoped to g *outside* of the origin of $\mathbb{R}^3$: Because the points $G(x)$ and $g(x)$ are ϵ-close, the straight line connecting them never passes through the origin. Pushing this straight line into the sphere, we obtain a homotopy through maps into S^2 between g and G, showing that G is a smooth representative of $[g]$.

Now, we are ready to sketch the Pontryagin-Thom construction: Let $p \in S^2$ be a regular value of g, which we assume to be smooth from now on. Then $g^{-1}(p)$ is a smooth, compact one-dimensional submanifold of S^3, hence a union of circles. For each connected component, the normal bundle is a rank 2 oriented bundle which can be thought of as a tubular neighborhood of the circle. For every $q \in g^{-1}(p)$, $D_q g$ induces an isomorphism $T_q S^3 / T_q(g^{-1}(p)) \cong T_p S^2$. This defines a trivialization of the normal bundle $\nu(g^{-1}(p))$ in S^3 (whose fibers are exactly $T_q S^3 / T_q(g^{-1}(p))$), since each fiber is identified with the same vector space $T_p S^2$. $g^{-1}(p)$ is called a *framed submanifold* (illustrated in figure 1).

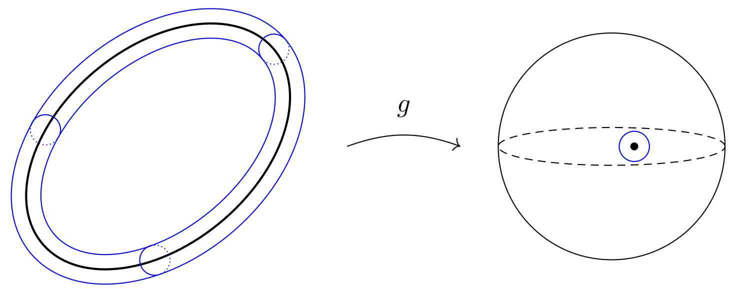


Figure 1: A framed circle in S^3

We would like to extract data that is independent of our initial choice of regular value. Pick a different regular value p' and connect the two by a path. The preimage of the path will, for a generic path, be an embedded surface (a bordism between $g^{-1}(p)$ and $g^{-1}(p')$) and the frame is also transferred, i.e. $g^{-1}(p)$ changes through a framed bordism (illustrated in figure 2). Thus, the framed bordism class is independent of our choice of regular value.

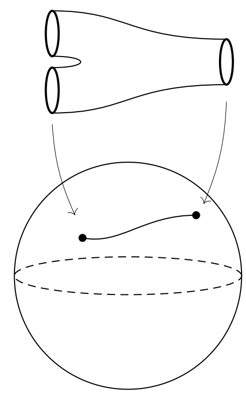


Figure 2: Picking two different regular values results in a framed cobordism between the preimages.

Now we consider homotopies of g: Let $H : S^3 \times I \to S^2$ be a homotopy from $g = H_0$ to $h = H_1$, which we can assume to be smooth (arguing as before). Pick a regular value p of g, h and H. Then $H^{-1}(p)$ is a surface in $S^3 \times I$ which projects under the canonical map $S^3 \times I \twoheadrightarrow S^3$ to a framed cobordism between $g^{-1}(p)$ and $h^{-1}(p)$. Thus, the framed submanifold $g^{-1}(p)$ only depends on $[g]$ up to framed bordism.

The next step is to reverse the process, i.e. determine $[g]$ from a framed bordism class of framed submanifolds of S^3. Given a framed 1-dimensional submanifold $K \subset S^3$ (which is a link in general), we identify an open tubular neighborhood $T \supset K$ with $K \times D^2$, using the framing. Now define $g : S^3 \to S^2$ as follows: Let $x \in T \cong K \times D^2$ and set $g(x) = \mathrm{proj}_2(x) \in D^2 = S^2 \setminus \{-p\}$. For any $x \notin T$, set $g(x) = -p \in S^2$. Then the preimage of p is K, g is smooth near K (and can be homotoped to be smooth everywhere), p is a regular value and the induced framing is the right one. This establishes a bijection

$$\pi_3(S^2) \longleftrightarrow \text{framed bordism classes of 1-dimensional submanifolds of } S^3$$

and completes the Thom-Pontryagin construction. It is a fact, which we do not prove, that every equivalence class can be represented by a connected, unknotted $S^1 \subset S^3$. All bordism classes are therefore only distinguished by framings. Two different framings of $S^1 \subset S^3$ differ by a map $\rho : S^1 \to SO(2)$—identify the fibers of $\nu(S^1)$, i.e. disks, over one point and then track how the two framings differ by a rotation at each point. But of course $SO(2) \cong S^1$, hence $\pi_3(S^2) \cong \pi_1(S^1) = \mathbb{Z}$. This finally allows us to discuss the possible homotopy types of M for $b_2(M) = 1$:

a) $g \simeq$ const. Then $M \simeq S^2 \vee S^4$ but this cannot be the homotopy type of any closed, oriented manifold: It does not satisfy Poincaré duality since the intersection form is degenerate.

b) If g corresponds to a generator of $\pi_3(S^2)$, the attaching map is the Hopf fibration. But this yields the standard description of the CW structure of $\mathbb{C}\mathrm{P}^2$, i.e. $M \simeq \mathbb{C}\mathrm{P}^2$ or $\overline{\mathbb{C}\mathrm{P}^2}$.

c) If $[g] = \lambda \in \mathbb{Z}$ with $|\lambda| \geq 2$ (λ is called the linking number), then $Q_M = (\pm\lambda)$, but this is not unimodular. Thus, this can never be the homotopy type of a closed, oriented manifold.

We conclude that for $m = 1$, there are precisely two possibilities (distinguishing orientations). Now, we turn to $b_2(M) = m \geq 2$. Then up to homotopy, we may take $g : S^3 \to \bigvee_j S_j^2$ to be smooth on the preimage of one hemisphere (not containing the "shared point" of the spheres) of each copy of S^2. Pick a regular value $p_j \in S_j^2$ and set $K_j := g^{-1}(p_j)$. This is again a framed 1-dimensional submanifold of S^3 and as before, $[g]$ corresponds to the framed bordism class of $K_1 \sqcup \cdots \sqcup K_m$.

This corresponds to an $(m \times m)$-matrix of linking numbers of the circles representing the K_j's (recall that up to framed bordism, we may take the K_j's to be unknotted circles). We will argue that this is the intersection form. every circle K_j bounds a surface $\Sigma_j \subset B^4$ (push a Seifert surface in S^3 into the 4-ball) and each Σ_j defines a closed surface by collapsing the boundary circle. In particular, when we glue onto $\bigvee_j S_j^2$, the boundary is collapsed, i.e. each Σ_j yields a closed surface $\tilde{\Sigma}_j$. These $\tilde{\Sigma}_j$'s yield a basis for $H_2(\bigvee_j S_j^2 \cup_g B^4)$: Since they intersect (only) the j-th copy of S^2 exactly in one point this is a "dual basis" to $\{S_j^2\}$. If one computes $\tilde{\Sigma}_i \cdot \tilde{\Sigma}_j$, it turns out to be the linking number $\mathrm{lk}(K_i, K_j)$. This shows that $(Q_M)_{ij} = \mathrm{lk}(K_i, K_j)$.

Requiring unimodularity in order to obtain the homotopy type of a manifold, we find all possibilities. Conversely, the intersection form tells us the linking numbers, which determine how the manifold is constructed and, in particular, the homotopy class $[g]$.

$\square$

One could get ambitious and ask if the intersection form actually determines the *homeomorphism type*. This turns out to be a much harder question, which as been partially answered in the affirmative by Freedman:

Theorem 3.22 (Freedman, 1982)**.**

- *Two simply connected, CCOS 4-manifolds are orientation-preserving homeomorphic if and only if their intersection forms are isomorphic over $\mathbb{Z}$.*
- *For every unimodular symmetric bilinear form over $\mathbb{Z}$, there is an oriented, simply connected, topological 4-manifold which realizes it as its intersection form.*

One particular corollary to these two theorems deserves a special mention:

Corollary 3.23 (Toplogical 4-dimensional Poincaré Conjecture)**.** *If a topological 4-manifold is homotopy equivalent to S^4 then it is homeomorphic to S^4.*

By way of summary of the results in this section, consider how one might go about determining the homeomorphism type of a given simply connected, CCOS 4-manifold M:

- If Q_M is definite, by Donaldson and Freedman's theorems, M is homeomorphic to a connected sum of $b_2(M)$ copies of either $\mathbb{C}\mathrm{P}^2$s or $\overline{\mathbb{C}\mathrm{P}}^2$s, corresponding to whether Q_M is positive or negative definite.
- If Q_M is indefinite and odd, then by Hasse-Minkowski and Freedman, M is homeomorphic to the manifold $b_2^+\mathbb{C}\mathrm{P}^2 \# b_2^- \overline{\mathbb{C}\mathrm{P}^2}$; if it is even, then it is homeomorphic to a topological manifold with intersection form $\frac{b_2(M)-|\sigma(M)|}{2} H + \frac{\sigma(M)}{8} E_8$.

3.3 Some Characteristic Classes

3.3.1 Euler Class and Second Stiefel-Whitney Class

Consider $E \to M$, an oriented rank 2 real vector bundle. Choose a metric h. We can choose fiberwise isometric, orientation-preserving local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^2$. On $U_i \cap U_j$, we have the transition functions

$$\begin{aligned} \psi_j \circ \psi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^2 &\to (U_i \cap U_j) \times \mathbb{R}^2 \\ (x, v) &\mapsto (x, g_{ji}(x)v) \end{aligned}$$

where $g_{ji} : U_{ij} \to \mathrm{SO}(2) = S^1$ is smooth. The g_{ij} satisfy the cocycle conditions:

- $g_{ij} = g_{ji}^{-1}$
- $g_{ij}g_{jk} = g_{ik}$ on U_{ijk}

and therefore define a cohomology class $[g_{**}]$ in $\check{H}^1(M; S_{S^1})$. This is independent of the choice of metric h. As done before, we use the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$ to induce a long exact sequence on the level of sheaf cohomology. Since $S_{\mathbb{R}}$ is a fine sheaf, we have an isomorphism $\delta : H^1(M; S_{S^1}) \cong H^2(M; S_{\mathbb{Z}})$.

Definition 3.24 (Euler Class)**.** We call $\delta[g_{**}] := e(E)$ the Euler class of E.

Remark 3.25. The ring homomorphism $\mathbb{Z} \hookrightarrow \mathbb{R}$ defines a map

$$\begin{aligned} H^2(M;\mathbb{Z}) &\longrightarrow H^2(M;\mathbb{R}) = H^2_{\mathrm{dR}}(M) \\ e(E) &\longmapsto e(E)_{\mathbb{R}} \end{aligned}$$

The latter can also be defined in terms of geometric quantities (relating to curvature).

It is clear that the Euler class classifies such bundles:

Proposition 3.26. *Two oriented rank 2 bundles $E,\ F \to M$ are orientation-preserving isomorphic if and only if $e(E) = e(F) \in H^2(M;\mathbb{Z})$.*

We list some fundamental properties of the Euler class:

- $e(E) = 0$ if and only if E is trivial.
- $e(\bar{E}) = -e(E)$.
- If $f : N \to M$ is an orientation-preserving smooth map and $E \to M$ a rank 2 oriented bundle, then $e(f^*E) = f^*e(E)$.

Proposition 3.27. *If M is a CCOS 4-manifold, then every $\alpha \in H_2(M;\mathbb{Z})$ is represented by a smoothly embedded surface.*

Proof. Let e be the Poincaré-dual of $\alpha \in H_2(M;\mathbb{Z})$ and $E \to M$ a smooth, oriented rank 2 vector bundle with $e(E) = e$. Let $s : M \to E$ be a smooth section that is transverse to the zero section $s_0(M) = M$. Thus, for every $p \in s(M) \cap s_0(M)$, $T_pM + T_ps(M) = T_pE$. Then the preimage $s^{-1}(0) = M \cap s(M)$ is a 2-dimensional smooth submanifold of M which inherits a natural orientation. It is a general fact about the Euler class that, given this setup, $\iota_*([S]) = \alpha \in H_2(M;\mathbb{Z})$. where $\iota : S \hookrightarrow M$ is the inclusion. Modulo torsion, this may be proven by showing that for any $\beta \in H_2(M;\mathbb{Z})$, $\alpha \cdot \beta = [S] \cdot \beta$ (but it holds true generally). ☐

Recall now that an oriented, real, rank 2 bundles E is equivalent to a complex line bundle L via the correspondence $SO(2) \cong U(1)$, i.e. $E \leftrightarrow L$, such that $L_{\mathbb{R}} = E$.

Definition 3.28 (First Chern Class)**.** We define $c_1(L) \coloneqq e(L_\mathbb{R})$ to be the first Chern class of a complex line bundle L, where $L_\mathbb{R}$ is oriented by the complex structure.

We can reformulate definition 2.40:

Definition 3.29 (Second Stiefel-Whitney Class)**.** Let $E, F \to M$ be an oriented, real vector bundles. Then there exists a unique $w_2(E) \in H^2(M;\mathbb{Z}_2)$ such that

(i) If E is trivial, then $w_2(E) = 0$.

(ii) $w_2(E \oplus F) = w_2(E) + w_2(F)$.

(iii) If E has rank 2, then $w_2(E) = r(e(E))$, where $r : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ is *reduction modulo 2*.

(iv) If $f : N \to M$ is a smooth map then $w_2(f^*(E)) = f^*(w_2(E))$.

These properties define the second Stiefel-Whitney class.

Uniqueness comes from uniqueness of $c_1(E)$, and we will not discuss existence (this is discussed at length in the book "Characteristic Classes" by Milnor & Stasheff). An additional property of w_2 is $w_2(E) = w_2(\bar{E})$.

Proposition 3.30. *If M is a CCOS 4-manifold and $w_2(TM) = 0$, then Q_M is even. The converse holds if $H_1(M;\mathbb{Z})$ is free of 2-torsion.*

Proof. Let $\iota : \Sigma \hookrightarrow M$ be a smoothly embedded, oriented surface representing a given class in $H_2(M;\mathbb{Z})$. Then $\iota^*TM = TM|_\Sigma = T\Sigma \oplus \nu(\Sigma)$ and both summands are oriented rank two bundles. Using the defining properties of the second Stiefel-Whitney class, we have:

$$\iota^* w_2(TM) = w_2(\iota^*TM) = w_2(\iota^*TM) = w_2(T\Sigma) + w_2(\nu(\Sigma))$$

Evaluating on $[\Sigma]$, we find

$$\langle \iota^* w_2(TM), [\Sigma]\rangle = r\langle e(T\Sigma), [\Sigma]\rangle + r\langle e(\nu(\Sigma)), [\Sigma]\rangle = r(\chi(\Sigma)) + r(\Sigma\cdot\Sigma) = r(\Sigma\cdot\Sigma)$$

where we used that the Euler class of the tangent bundle evaluates on the fundamental class to the Euler characteristic $\chi(\Sigma) = 2 - 2g \equiv 0 \mod 2$ while the normal bundle of Σ can be viewed as a tubular neighborhood, hence the zero locus of a generic section is exactly the self-intersection of Σ.

The equation $\langle \iota^* w_2(TM), [\Sigma]\rangle = r(\Sigma\cdot\Sigma)$ makes it clear that if $w_2(TM) = 0$, the intersection form must be even, since every class is represented by an embedded surface. Conversely, if Q_M is even we see that $\langle \iota^* w_2(TM), [\Sigma]\rangle = 0$ for every embedded surface Σ. Using the universal coefficients theorem, the Ext-term vanishes if there is no 2-torsion, hence in this case we conclude that $w_2(TM) = 0$. $\square$

Corollary 3.31. *For any class $\alpha \in H_2(M;\mathbb{Z})$, $\langle w_2(TM), \alpha\rangle \equiv \alpha\cdot\alpha \mod 2$.*

In particular, if M is simply connected, $H_*(M;\mathbb{Z})$ is torsion-free so by the results of section 2.3.2 we have:

Corollary 3.32. *Let M be a CCOS, simply connected 4-manifold. Then M is* Spin *if and only if Q_M is even.*

The following important result speaks to the existence of Spin^c structures:

Theorem 3.33 (Whitney)**.** *For any CCOS 4-manifold M, there exists some $c \in H^2(M;\mathbb{Z})$ such that $r(c) = w_2(TM)$, where r is reduction modulo 2. Hence such manifolds always admit a Spin^c structure.*

Proof. By naturality of the universal coefficients theorem under reduction, we have a commutative ladder

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}) \arrow[r, "f"] \arrow[d, "r"] & H^2(M;\mathbb{Z}) \arrow[r, "g"] \arrow[d, "r"] & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2) \arrow[r, "f'"'] & H^2(M;\mathbb{Z}_2) \arrow[r, "g'"'] & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2) \arrow[r] & 0
\end{tikzcd}$$

An element $\varphi \in \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2)$ lifts under r if and only if $\varphi(t) = 0$ for every torsion element $t \in H_2(M;\mathbb{Z})$. The bilinearity of the intersection form guarantees that it kills all torsion. Indeed, if α were k-torsion but $\alpha \cdot \alpha \neq 0$, we would have $(k\alpha)\cdot(k\alpha) = 0 = k^2(\alpha\cdot\alpha) \neq 0$. Since $Q_M(\alpha,\alpha) \equiv \langle w_2(TM), \alpha\rangle \mod 2$, we see that they agree as elements of $\mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2)$, i.e. $g'(w_2(TM)) = \langle w_2(TM), -\rangle =: \omega$ must lift.

By surjectivity of the top-right horizontal arrow, there exists some $x \in H^2(M;\mathbb{Z})$ such that $r(g(x)) = \omega$. Commutativity of the square tells us that $g'(r(x)) = \omega = g'(w_2(TM))$. But then exactness of the bottom row tells us that there exists some $\gamma \in \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$ such that $f'(\gamma) = r(x) - w_2(TM)$. The first vertical map is surjective by general homological algebra arguments, hence there is some $\kappa \in \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z})$ such that $r(\kappa) = \gamma$, hence $f'(\gamma) = f(r(\kappa)) = r(f(\kappa))$. Now set $c = x - f(\kappa)$. Then $r(c) = r(x) - r(f(\kappa)) = r(x) - f(\gamma) = w_2(TM)$, hence c is an integral lift of $w_2(TM)$. ☐

3.3.2 Chern Classes

Definition 3.34 (Chern Classes)**.** If L is a complex line bundle, the *total Chern class* of L is $c(L) = 1 + c_1(L)$. If $E = L_1 \oplus L_2 \oplus \ldots \oplus L_k$ is a direct sum of complex line bundles, we extend the above definition in the obvious way:

$$c(E) := (1 + c_1(L_1)) \smile (1 + c_1(L_2)) \smile \ldots \smile (1 + c_1(L_k))$$

Expanding the above, we obtain the *Chern classes* $c_i(E)$:

$$c(E) = 1 + \underbrace{\sum_{i=1}^{k} c_1(L_i)}_{c_1(E)\in H^2(M;\mathbb{Z})} + \underbrace{\sum_{1\le i<j\le k}^{k} c_1(L_i) \smile c_1(L_j)}_{c_2(E)\in H^4(M;\mathbb{Z})} + \ldots + \underbrace{\prod_{i=1}^{k} c_1(L_i)}_{c_k(E)\in H^{2k}(M;\mathbb{Z})}$$

The following proposition gives us a way to generalize the definition to arbitrary complex vector bundles:

Proposition 3.35. *For every complex vector bundle $E \to M$, there exists a so-called "splitting manifold" $f : N \to M$ with the following properties*

(i) $f^*E \cong L_1 \oplus \ldots \oplus L_k$*, where the L_i are line bundles.*

(ii) f^* *is injective on* $H^*(M;\mathbb{Z})$*.*

Sketch of Proof. Set $n = \mathrm{rank}_{\mathbb{C}} E$ and consider the *projectivized bundle* $\pi : \mathbb{P}(E) \to M$, which is the $\mathbb{C}\mathrm{P}^{n-1}$-bundle over M with $(\mathbb{P}(E))_p = \mathbb{P}(E_p)$. The transition functions act on fibers by the action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}\mathrm{P}^{n-1}$ (the action descends from the action on $\mathbb{C}^n$). Now consider the pullback bundle π^*E, which yields the following diagram:

$$\begin{tikzcd}
L_1 \subset \pi^*E \arrow[r] \arrow[d] \arrow[d, shift right=6] & E \arrow[d, "p"] \\
\mathbb{P}(E) \arrow[r, "\pi"'] & M
\end{tikzcd}$$

where L_1 is the tautological line bundle

$$L = \{(\ell, v) \in \mathbb{P}(E) \times E \mid v \in \ell\} \subset \{(\ell, v) \in \mathbb{P}(E) \times E \mid \pi(\ell) = p(v)\} = \pi^*E$$

Then $\pi^*E \cong L_1 \oplus Q$ where Q is a complement $Q \cong \pi^*E/L_1$. Iterating this process, we get a tower of projectivizations such that eventually $f^*E \cong \bigoplus_j L_j$. Injectivity on the level of cohomology follows from the Leray-Hirsch theorem. ☐

Definition 3.36. This allows us to define $c(E)$ for an arbitrary complex vector bundle as unique element of $H^*(M;\mathbb{Z})$ that maps to $c(f^*E) = c(L_1 \oplus \dots \oplus L_k)$ under f^*.

Remark 3.37. Of course, one should really check that using or deriving identities involving Chern classes does not take one out of the image of f^*. This can be done inductively by carefully using the (proof of the) Leray-Hirsch theorem.

This method of defining Chern classes in terms of split vector bundles and the techniques that it enables one to make use of collectively embody the so-called *splitting principle*. The basic properties of the Chern classes are:

(i) If $E \to M$ is trivial, then $c_i(E) = 0$ for all $i > 0$.

(ii) $c(E \oplus F) = c(E) \cdot c(F)$.

(iii) $w_2(E_{\mathbb{R}}) = r(c_1(E))$.

(iv) $c_i(f^*E) = f^*c_i(E)$ for all i.

(v) $c_i(\bar{E}) = (-1)^i c_i(E)$.

(vi) $c_i(E) = 0$ for $i > \operatorname{rank}_{\mathbb{C}} E$.

(vii) $c_i(E) = e(E_{\mathbb{R}})$ for $i = \operatorname{rank}_{\mathbb{C}} E$.

Note that we omit the cup product in our notation: Since Chern classes commute, one can think of the cup product of multiplication of polynomials. We will use the standard notation $e(M) = e(TM)$, and $w_2(M) = w_2(TM)$ from now on. If M is an almost complex manifold, $c_i(M) = c_i(TM)$. However, note that the Chern classes of M depend on the choice of an almost complex structure (so one should properly write $c_i(M, J)$), but confusion rarely arises.

Lemma 3.38. *$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1} = 0)$, where $x \in H^2(M;\mathbb{Z})$ is of degree two, and $\mathbb{Z}[x]/(x^{n+1} = 0) = \{a_0 + a_1x^1 + a_2x^2 + \dots a_nx^n| \mid a_i \in \mathbb{Z}\}$ is the polynomial ring over x. Hence,*

$$c(\mathbb{CP}^n) = (1+x)^{n+1} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k = \sum_{k=0}^{n} c_k(\mathbb{CP}^n)$$

Proof. The first statement is proven using cellular homology for the additive structure, while the multiplicative structure is determined as follows: The standard embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ yields the positive generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$ and the Poincaré dual $[\mathbb{CP}^1]$ self-intersects once. Thus, the cup product of the generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$ with itself yields the positive generator of $H^4(\mathbb{CP}^2;\mathbb{Z})$. Proceeding inductively along those lines yields the claim.

For the second part, let L be the tautological line bundle over $\mathbb{CP}^n$. Then by linear algebra arguments, $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \bigoplus_{j=0}^n L$. Hence $c(T\mathbb{CP}^n) = (1+x)^{n+1}$. ◻

Example 3.39 ($M = \mathbb{CP}^2$)**.** From the above formula, $c_1 = 3x$, $c_2 = 3x^2$. Since $T\mathbb{CP}^2$ is of complex dimension 2, $c_2(\mathbb{CP}^2) = e(\mathbb{CP}^2)$, i.e.

$$\langle c_2(\mathbb{CP}^2), [\mathbb{CP}^2] \rangle = \langle e(\mathbb{CP}^2), [\mathbb{CP}^2] \rangle = \chi(\mathbb{CP}^2) = 3 = b_0 + b_2 + b_4$$

The final result in this section is concerned with surfaces embedded in almost complex 4-manifolds:

Theorem 3.40 (Adjunction Formula)**.** *Let M be an oriented, smooth 4-manifold with an almost complex structure*[7] *J compatible with the orientation. Let $\iota : \Sigma \to M$ be a smoothly embedded surface with $J(T\Sigma) = T\Sigma$, i.e. Σ is an*

[7] A $J \in \Gamma(\text{End } TM)$ such that $J_p^2 = -\text{Id}_{T_pM}$ for each $p \in M$ is said to be an almost complex structure for TM.

almost complex submanifold. Then the genus of Σ is given by

$$g(\Sigma) = 1 + \frac{1}{2}\Big(\Sigma \cdot \Sigma - \langle c_1(M), \iota_*[\Sigma]\rangle\Big)$$

Proof. Since Σ is a J-holomorphic submanifold, we know that as complex vector bundles

$$TM|_\Sigma = T\Sigma \oplus \nu(\Sigma)$$

Hence, we have

$$\iota^* c_1(M) = c_1\left(TM|_\Sigma\right) = c_1(T\Sigma) + c_1(\nu(\Sigma)) = e(T\Sigma) + e(\nu(\Sigma))$$

whence we can compute

$$\langle c_1(M), \iota_*[\Sigma]\rangle = \chi(\Sigma) + \Sigma \cdot \Sigma = 2 - 2g(\Sigma) + \Sigma \cdot \Sigma$$

□

In particular, we see that the genus is determined by the homology class.

Example 3.41. A holomorphic curve of degree d, $\Sigma_d \subset \mathbb{C}\mathrm{P}^2$, is a smooth holomorphic curve of degree d, i.e. $[\Sigma_d] = d \cdot [\mathbb{C}\mathrm{P}^1] \in H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$, where $[\mathbb{C}\mathrm{P}^1]$ is the generator of $H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. Therefore, $\Sigma \cdot \Sigma = d^2$ and by the adjunction formula,

$$g(\Sigma_d) = 1 + \frac{1}{2}(d^2 - 3\langle x, [\Sigma_d]\rangle) = \frac{1}{2}(d^2 - 3d + 2) = \frac{1}{2}(d-1)(d-2)$$

This is known as the *degree formula*.

3.3.3 Pontryagin Classes

Definition 3.42 (Pontryagin Classes)**.** Let $V \to M$ be a real vector bundle. We define

$$p_i(V) := (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M;\mathbb{Z})$$

the Pontryagin classes of V. The *total Pontryagin class* of V is $p(V) = \sum_j p_j(V)$.

The Pontryagin classes inherit all the properties of the Chern classes. Moreover, note that $\operatorname{rank}_{\mathbb{R}} V = \operatorname{rank}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$, hence $p_i = 0$ if $2i > \operatorname{rank}_{\mathbb{R}} V$.

Example 3.43.

(i) $p_i(V) = 0$ for all i if $\operatorname{rank}_{\mathbb{R}} V = 1$.

(ii) Assume that $\operatorname{rank}_{\mathbb{R}} V = 2$. Then $p_i(V) = 0$ for $i \geq 2$, and $p_1(V) = -c_2(V \otimes_{\mathbb{R}} \mathbb{C})$. If V is orientable, we fix an orientation and henceforth think of V as a complex line bundle L. For a complex vector bundle E, $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$. Then

$$\begin{aligned} c_{2i}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) &= c_{2i}(E \oplus \bar{E}) = c_{2i}(E) + c_{2i-1}(E)c_1(\bar{E}) + \ldots + c_1(E)c_{2i-1}(\bar{E}) + c_{2i}(\bar{E}) \\ &= c_{2i}(E) - c_{2i-1}(E)c_1(E) + c_{2i-2}(E)c_2(E) - \cdots - c_1(E)c_{2i-1}(E) + c_{2i}(E) \end{aligned}$$

Hence, in the case where $E = V$ is orientable and of real-rank 2, we fix an orientation and think of V as a complex line bundle L. $c_1(L)$ is then defined, and we have the important relation

$$p_1(V) = -c_2(L \oplus \bar{L}) = c_1^2(L) = e^2(V)$$

Theorem 3.44 (Signature Formula, Thom-Hirzebruch)**.** *For a CCOS 4-manifold M, the signature is given by*

$$\sigma(M) = \frac{1}{3}\langle p_1(TM), [M]\rangle$$

Corollary 3.45.

(i) $\sigma(M) = 0$ *if and only if* $p_1(TM) = 0$.

(ii) $p_1(M)$ *is a multiple of 3 since* $\sigma(M) \in \mathbb{Z}$.

Example 3.46 (Surfaces in $\mathbb{C}\mathrm{P}^3$)**.** We discuss the analog of holomorphic curves in $\mathbb{C}\mathrm{P}^2$ in one dimension higher, namely algebraic surfaces in $\mathbb{C}\mathrm{P}^3$: Consider $\iota : X_d \hookrightarrow \mathbb{C}\mathrm{P}^3$, a smooth algebraic surface of degree d: X_d is the zero locus of a generic homogeneous polynomial of degree d on $\mathbb{C}^4$. Let $x \in H^2(\mathbb{C}\mathrm{P}^3, \mathbb{Z})$ be the positive generator $\langle x, [\mathbb{C}\mathrm{P}^1]\rangle = 1$. As before, we have $T\mathbb{C}\mathrm{P}^3|_{X_d} = TX_d \oplus \nu(X_d)$, since X_d is a holomorphic submanifold. Then

$$\begin{aligned} &\iota^* c(T\mathbb{C}\mathrm{P}^3) = c(TX_d) \smile c(\nu(X_d)) \\ \implies\ &\iota^*(1+x)^4 = (1 + c_1(TX_d) + c_2(TX_d))(1 + c_1(\nu(X_d))) \end{aligned}$$

Equating the polynomials degree-by-degree, we see that

$$\begin{aligned} \iota^*(4x) &= c_1(TX_d) + c_1(\nu(X_d)) \\ \iota^*(6x^2) &= c_2(TX_d) + c_1(TX_d)c_1(\nu(X_d)) \end{aligned}$$

Now, it is a fact from complex geometry that that $\nu(X_d) = \mathcal{L} = \mathcal{O}(d)|_{X_d}$, where $\mathcal{O}(d) \to \mathbb{C}\mathrm{P}^3$ is the holomorphic line bundle with $c_1(\mathcal{L}) = d \cdot x$. Hence,

$$\begin{aligned} &\iota^*(4x) = c_1(TX_d) + c_1(\iota^*\mathcal{L}) = c_1(TX_d) + \iota^*(d \cdot x) \\ \implies\ &c_1(TX_d) = \iota^*((4-d)x) = (4-d)\iota^* x \end{aligned}$$

Using this in the equation for c_2, we get

$$c_2(TX_d) = (d(d-4)+6)\iota^*(x^2)$$

We can now compute the Euler characteristic:

$$\begin{aligned} \chi(X_d) &= \langle c_2(TX_d), [X_d]\rangle = (d(d-4)+6)\langle \iota^* x^2, [X_d]\rangle \\ &= d(d^2 - 4d + 6) \end{aligned}$$

where we used that $[X_d] = d[\mathbb{C}\mathrm{P}^2]$. The first Pontryagin class is now easy to compute:

$$\begin{aligned} p_1(TX_d) &= -c_2(TX_d \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(TX_d \oplus \overline{TX}_d) \\ &= -2c_2(TX_d) + c_1^2(TX) = \iota^*((-2(d(d-4)+6)) + (4-d)^2)x^2) \\ &= (4-d^2)\iota^* x^2 \end{aligned}$$

By the signature formula,

$$\sigma(X_d) = \frac{1}{3}(4-d^2)\langle \iota^* x^2, [X_d]\rangle = \frac{1}{3}d(4-d^2)$$

Note that this is indeed an integer: If $d \equiv 0 \mod 3$ this is clear, while if $d \equiv \pm 1 \mod 3$ then $d^2 = 1 \mod 3$, hence $4 - d^2 \equiv 0 \mod 3$. To complete our analysis we use the *Lefschetz hyperplane theorem*, which implies that $\pi_1(X_d) = 1$. This implies that $b_1(X_d) = 0 = b_3(X_d)$, which in turn implies that $b_2(X_d) = \chi(X_d) - 2$. Since we also know $\sigma(M)$, we can now determine $b_2^{\pm}(M)$:

$$b_2^{\pm}(X_d) = \frac{1}{2}(b_2(X_d) \pm \sigma(X_d)) = \frac{1}{2}(\chi(X_d) - 2 \pm \sigma(M)) = \frac{d}{2}\Big(d^2 - 4d + 6 \pm \frac{1}{3}(4-d^2)\Big) - 1$$

Let us investigate the situation for low values of d:

(i) $d = 1$ yields $\mathbb{C}\mathrm{P}^2$; $\chi(X_1) = 3$ and $\sigma(X_1) = 1$.

(ii) For $d = 2$, $\chi(X_2) = 4$ and $\sigma(X_2) = 0$.

Lemma 3.47. *X_2 is diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 = S^2 \times S^2$.*

Proof. Consider the so-called *Segre map*

$$\begin{aligned} f : \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 &\longrightarrow \mathbb{C}\mathrm{P}^3 \\ ([x:y],[z:w]) &\longmapsto (xz : xw : yz : yw) \end{aligned}$$

This is a well-defined holomorphic map, injective and in fact an immersion, hence an embedding since it is automatically proper (this is proven in exercise 6.2). It is also precisely the zero-locus of the homogeneous second degree polynomial $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f(t_0, t_1, t_2, t_3) = t_0 t_3 - t_1 t_2$. Any other (generic) degree 2 polynomial can be deformed to it, hence $X_2 \cong S^2 \times S^2$. $\square$

(iii) For $d = 3$ we have $\chi(X_3) = 9$ and $\sigma(M) = -5$. The Hasse-Minkowski classification and Freedman's theorem tell us that this manifold is homeomorphic to $\mathbb{C}\mathrm{P}^2 \# 6\overline{\mathbb{C}\mathrm{P}^2}$. In fact, it is diffeomorphic to this space, but we will not prove that.

(iv) $d = 4$ yields $\chi(X_4) = 24$ and $\sigma(M) = -16$. By the Hasse-Minkowski classification, the intersection form is determined by the parity of Q_M. Since there is no torsion in this case, this is determined by $w_2(X_d) = r(c_1(X_d)) = r((4-d)\iota^* x) \equiv d\,\iota^* x \mod 2$. Hence for $d = 4$, Q_M is even and $Q_M = 3H \oplus 2E_8$ by Hasse-Minkowski.

Definition 3.48 ($K3$ Surface)**.** A $K3$ surface X is a compact, complex surface with $\pi_1(X) = 0$ and $c_1(X) = 0$.

It is a fact, though not easy to prove, that any such spaces are diffeomorphic. Observe that a $K3$ surface saturates the $\frac{11}{8}$ inequality.

We summarize these results in a table:

	$d=1$	$d=2$	$d=3$	$d=4$
$\chi(X_d)$	3	4	9	24
$\sigma(X_d)$	1	0	-5	-16
Diffeomorphic to	$\mathbb{C}\mathrm{P}^2$	$S^2 \times S^2$	$\mathbb{C}\mathrm{P}^2 \# 6\overline{\mathbb{C}\mathrm{P}^2}$	$K3$

Regarding the $\frac{11}{8}$-conjecture, we can say a little bit more:

Proposition 3.49. *The $\frac{11}{8}$-conjecture is equivalent to the following statement: Every simply connected CCOS 4-manifold with even intersection form is homeomorphic to a connected sum of copies of $K3$, $\overline{K3}$ and $S^2 \times S^2$.*

Proof. Suppose that our 4-manifold M is homeomorphic to a connected sum of copies of $K3$, $\overline{K3}$ and $S^2 \times S^2$, i.e. $M = a \# b\overline{K3} \# c(S^2 \times S^2)$. Then

$$\begin{aligned} b_2(M) &= 22(a+b) + 2c \\ \sigma(M) &= 16(b-a) \end{aligned}$$

So we compute

$$\frac{11}{8}|\sigma(M)| = \frac{11}{8}|16(b-a)| \leq \frac{11}{8}16(b+a) = 22(a+b) \leq b_2(M)$$

as desired. Conversely, suppose that M satisfies the 11/8-conjecture. Since M is Spin, Rohlin's theorem (for more information, see theorem 4.6) asserts that 16 divides $\sigma(M)$, so we can write $\sigma(M) = 16b$ for some $b \in \mathbb{Z}$. Note that we never need both $K3$ and $\overline{K3}$ since by Freedman, $K3 \# \overline{K3} \cong 22(S^2 \times S^2)$. There are three cases.

- If $b = 0$, hence $\sigma(M) = 0$. Then Hasse-Minkowski, tells us that $Q_M \cong aH$ for some $\mathbb{Z} \ni a \geq 1$. Freedman's theorem then guarantees that $M \cong a(S^2 \times S^2)$.
- If $b < 0$, then we want to show that $M \cong |b|K3\#a(S^2 \times S^2)$, where b is determined by $\sigma(M)$, and $a = (b_2(M) - 22|b|)/2$. For this to make sense, we need to check that $a \geq 0$. First of all, since Q_M is even, b_2 is even (by Hasse-Minkowski), so a is certainly an integer. Then

$$b_2(M) \geq \frac{11}{8}|\sigma(M)| = 22|b| \implies b_2(M) - 22|b| \geq 0 \implies a \geq 0$$

- Finally, if $b > 0$, by similar arguments, $M \cong b\overline{K3}\#a(S^2 \times S^2)$, where $a = (b_2(M) - 22b)/2 \geq 0$ (again, by the 11/8-conjecture). This completes the proof.

□

3.4 Self-Duality and the Half-de Rham Complex

3.4.1 Hodge Decomposition

Let V be an oriented vector space with of dimension 4 equipped with a scalar product $\langle\cdot,\cdot\rangle$, i.e. the structure of a tangent space of an oriented, Riemannian manifold.

Definition 3.50 (Hodge Star Operator)**.** We define the Hodge Star operator $* : \Lambda^k(V^*) \to \Lambda^{4-k}(V^*)$ by

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \mathrm{vol}$$

where we use the induced inner product on forms to make sense out of $\langle\alpha, \beta\rangle$.

It satisfies $*^2 = \mathrm{Id}$ on Λ^2V^*, so it has eigenvalues ± 1. Hence $\Lambda^2V^* = \Lambda^2_+V^* \oplus \Lambda^2_-V^*$.

Definition 3.51. $\Lambda^2_\pm(V^*)$ are the space of *self-dual* (resp. *anti self-dual*) 2-forms.

Recall that given an oriented orthonormal basis, $\{e_0, \dots e_3\}$, we have the following basis for $\Lambda^2_\pm(V^*)$:

$$\begin{gathered} e_0 \wedge e_1 \pm e_2 \wedge e_3 \\ e_0 \wedge e_2 \mp e_1 \wedge e_3 \\ e_0 \wedge e_3 \pm e_1 \wedge e_2 \end{gathered}$$

Now let X be an oriented Riemannian 4-manifold with a metric g. We then have the decompositions

$$\begin{aligned} \Lambda^2T^*X &= \Lambda^2_+T^*X \oplus \Lambda^2_-T^*X \\ \implies \Omega^2(TX) &= \Omega^2_+(X) \oplus \Omega^2_-(X) \end{aligned}$$

Recall the L^2 inner product of forms (here, we start assuming X is closed):

$$\langle\alpha, \beta\rangle_{L^2} = \int_X g(\alpha, \beta)\mathrm{vol}_g$$

where we have once again induced $g(-,-)$ on forms.

Definition 3.52 (Laplace Operator)**.** We define the Laplace operator of g as $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, where $\mathrm{d}^* : \Omega^k(X) \to \Omega^{k-1}(X)$. is the formal adjoint of d with respect to the L^2 scalar product, $\mathrm{d}^* = \pm * \mathrm{d} *$, where the sign depends on the dimension and degree of the form it acts on. A form $\alpha \in H^*(X;\mathbb{Z})$ is called *harmonic* if $\Delta\alpha = 0$ and the space of harmonic k-forms is denoted by $\mathcal{H}^k(X)$.

Lemma 3.53. *If X is closed, then for a 2-form α, $\Delta\alpha = 0$ if and only if $\mathrm{d}\alpha = 0 = \mathrm{d}^*\alpha$.*

Proof. We simply use that d and d^* are each other's adjoints:

$$\int_X g(\Delta\alpha, \alpha)\mathrm{vol}_g = \int_X \left(|\mathrm{d}\alpha|^2 + |\mathrm{d}^*\alpha|^2\right) \mathrm{vol}_g$$

This shows the equivalence. □

Hence, every harmonic form α on a closed Riemannian manifold is closed. Therefore there is a canonical mapping $\mathcal{H}^i(X) \to H^i_{\mathrm{dR}}$.

Theorem 3.54 (Hodge)**.** *Every de Rham cohomology class contains a unique harmonic representative:* $H^k_{dR}(X) \cong \mathcal{H}^k(X)$ *and the isomorphism is given by the projection* $\mathcal{H}^i(X) \to H^i_{dR}(M)$*. Moreover, there is an orthogonal decomposition*

$$\Omega^k(X) = \mathrm{d}(\Omega^{k-1}(X)) \oplus \mathcal{H}^k(X) \oplus \mathrm{d}^*(\Omega^{k+1})$$

Notice that $*$ maps $\mathcal{H}^i(X)$ to $\mathcal{H}^{4-i}(X)$, and as above, we have the decomposition of the harmonic 2-forms into the space of self-dual and anti self-dual harmonic 2-forms:

$$\mathcal{H}^2(X) = \mathcal{H}^2_+(X) \oplus \mathcal{H}^2_-(X)$$

Assume that α is closed and (anti) self-dual. Then first observe that $\mathrm{d}^*\alpha = \pm * \mathrm{d} * \alpha = \pm * \mathrm{d}\alpha = 0$. Furthermore, we have:

$$Q_X(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \pm \int_X \alpha \wedge *\alpha = \pm \int_X |\alpha|^2 \mathrm{vol}_g \gtrless 0$$

with equality if and only if $\alpha \equiv 0$. Therefore $b_2^\pm(X) = \dim \mathcal{H}^2_\pm$. Now consider α self-dual and β anti self-dual. Then

$$\begin{aligned} Q_X(\alpha, \beta) = \int_X \alpha \wedge \beta = \int_X (*\alpha) \wedge \beta = \int_X \beta \wedge *\alpha = \int_X g(\alpha, \beta)\mathrm{vol}_g \\ = -\int_X \alpha \wedge *\beta = -\int_X g(\alpha, \beta)\mathrm{vol}_g \end{aligned}$$

Hence $Q_X(\alpha, \beta) = 0$. Thus, the decomposition $\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$ is orthogonal with respect to Q_X (which coincides with the L^2-inner product). In fact, the splitting is even orthogonal with respect to the pointwise metric induced by g.

3.4.2 The Half-de Rham Complex

Now, consider the de Rahm complex of X^n:

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(X) \longrightarrow 0$$

Proposition 3.55. *For a closed, oriented, smooth 4-manifold X, the following is a complex with finite-dimensional cohomology:*

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{d^+} \Omega^2_+(X) \longrightarrow 0$$

It is sometimes called the half-de Rham complex. Note that d^+ is the composition $\Omega^1(X) \xrightarrow{\mathrm{d}} \Omega^2(X) \xrightarrow{\pi^+} \Omega^2_+(X)$. The alternating sum of dimensions of the cohomology is $\frac{1}{2}\big(\chi(X) + \sigma(X)\big)$.

Proof. Clearly $\mathrm{d}^+ \circ \mathrm{d} = \pi^+ \circ \mathrm{d}^2 = 0$ and $H^0_{\mathrm{dR}}(X)$ is the first cohomology vector space of the complex. Let $\alpha \in \Omega^1(X)$ lie in the kernel of d^+. Now observe

$$\begin{aligned} 0 = \int_X \mathrm{d}(\alpha \wedge \mathrm{d}\alpha) &= \int_X \mathrm{d}\alpha \wedge \mathrm{d}\alpha = \int_X (\mathrm{d}^+\alpha + \mathrm{d}^-\alpha) \wedge (\mathrm{d}^+\alpha + \mathrm{d}^-\alpha) \\ &= \int_X \mathrm{d}^+\alpha \wedge *\mathrm{d}^+\alpha - \mathrm{d}^-\alpha \wedge *\mathrm{d}^-\alpha \\ &= \int_X (|\mathrm{d}^+\alpha|^2 - |\mathrm{d}^-\alpha|^2)\mathrm{vol}_g \end{aligned}$$

We see that $\mathrm{d}^+\alpha = 0$ if and only if $\mathrm{d}^-\alpha = 0$, which is then equivalent to $\mathrm{d}\alpha = 0$. Hence the middle cohomology vector space is simply $H^1_{\mathrm{dR}}(X)$. For the last, we just need to find $\operatorname{coker} \mathrm{d}^+$. Take $h \in \mathcal{H}^2_+(X)$ and $\alpha \in \Omega^1(X)$. Then

$$\int_X h \wedge \mathrm{d}^+\alpha = \int_X \mathrm{d}^+\alpha \wedge *h = \langle \mathrm{d}^+\alpha, h\rangle_{L^2}$$

but on the other hand

$$\int_X h \wedge \mathrm{d}^+\alpha = \int_X \mathrm{d}(h \wedge \alpha) = 0$$

by Stokes' theorem, plus the fact that $\Omega^2_+(X) \oplus \Omega^2_-(X)$ is an orthogonal decomposition. Hence the image of $\Omega^1(X)$ under d^+ is orthogonal to $\mathcal{H}^2_+(X)$. Now, we use Hodge decomposition (theorem 3.54) to uniquely write $\omega \in \Omega^2_+$ as $\omega = h + \mathrm{d}\alpha + \mathrm{d}^*\beta$ where h is harmonic. By uniqueness of Hodge decomposition and self-duality, we see $*h = h$, $*\mathrm{d}\alpha = \mathrm{d}^*\beta$ and $*\mathrm{d}^*\beta = \mathrm{d}\alpha$, i.e. $\omega = h + \mathrm{d}\alpha + *\mathrm{d}\alpha = h + 2\mathrm{d}^+\alpha$. But then clearly $\Omega^2_+(X)/\mathrm{d}^+(\Omega^1(X)) \cong \mathcal{H}^2_+(X)$, which is our third cohomology vector space. The alternating sum of the dimensions is:

$$b_0(X) - b_1(X) + b_2^+(X) = \frac{1}{2}\chi(X) + \frac{1}{2}b_2^+(X) - \frac{1}{2}b_2^-(X) = \frac{1}{2}(\chi(X) + \sigma(X))$$

as claimed. □

We can "roll up" all the information about this complex into a single invariant. Consider the operator $\mathrm{d}^+ \oplus \mathrm{d}^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X)$.

Definition 3.56. The *Fredholm index* of $\mathrm{d}^+ \oplus \mathrm{d}^*$ is defined as

$$\begin{aligned} \operatorname{ind}(\mathrm{d}^+ \oplus \mathrm{d}^*) &= \dim\ker(\mathrm{d}^+ \oplus \mathrm{d}^*) - \dim\operatorname{coker}(\mathrm{d}^+ \oplus \mathrm{d}^*) \\ &= b_1(X) - b_2^+(X) - b_0(X) \end{aligned}$$

since $\ker(\mathrm{d}^+ \oplus \mathrm{d}^*) = (\mathcal{H}^1(X) \oplus \mathrm{d}^*\Omega^2(X)) \cap (\mathcal{H}^1(X) \oplus \mathrm{d}\Omega^0(X)) = \mathcal{H}^1(X)$. Here, we used the fact that an operator with finite-dimensional kernel and cokernel is Fredholm to make the index well-defined.

4 The Dirac Operator and the Seiberg-Witten Equations

4.1 Elliptic Operators

Let $E,\ F \to M$ be bundles, and $P : \Gamma(E) \to \Gamma(F)$ a first-order differential operator (think of P as expressed in terms of a covariant derivative, using only one power).

Definition 4.1. The *symbol* of P is a bundle map $\sigma(P) : T^*M \to \mathrm{Hom}(E, F)$, defined as follows. Let $\xi \in T_p^*M$, and $e \in E_p$. Choose an extension $\tilde{e}$ of e to a section of E, and choose a smooth function $f \in C^\infty(M)$ on M with $f(p) = 0$ and $(\mathrm{d}f)_p = \xi$ (e.g. multiply a representative of the germ which satisfies these conditions with a cutoff function). Then

$$(\sigma(P)(\xi))(e) := (P(f \cdot \tilde{e}))(p)$$

We will not show that this is indeed well-defined, but suffice it to say that it is crucial that P is a local operator.

Definition 4.2 (Elliptic Operator)**.** P is *elliptic* if $\sigma(P)(\xi) \in \mathrm{Hom}(E, F)$ is an isomorphism for any $\xi \neq 0$.

If P is elliptic, it is also *Fredholm*, that is, $\dim\ker P$ and $\dim\operatorname{coker} P$ are both finite. In particular, the Fredholm index $\operatorname{ind} P := \dim\ker P - \dim\operatorname{coker} P$ is well-defined. Observe that, P can only be elliptic if $\operatorname{rank} E = \operatorname{rank} F$.

Example 4.3.

(i) $P = \mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. As in the definition above, pick an $\omega \in \Lambda^k(T_p^*(M))$, and extend it to a form $\tilde{\omega} \in \Omega^k(M)$. Choose a function f such that $f(p) = 0$ and $(\mathrm{d}f)(p) = \xi$ for a given $\xi \in T_p^*M$. Then

$$\mathrm{d}(f \cdot \tilde{\omega})(p) = (\mathrm{d}f \wedge \tilde{\omega} + f\mathrm{d}\tilde{\omega}(p) = (\xi \wedge \omega)(p)$$

Hence, $\sigma(\mathrm{d})(\xi) = \xi\wedge$. But this map is not invertible, since e.g. multiples of ξ are in its kernel. Hence, d is not an elliptic operator. However, we see that if we take a direct sum of (non-elliptic) operators *whose symbols have non-overlapping kernel*, we can obtain an elliptic operator: This is exactly what we did with $\mathrm{d}^+ \oplus \mathrm{d}^*$.

(ii) $P = D_A : \Gamma(V) \to \Gamma(V)$, the Dirac operator of a Spin^c-structure with Spin^c-connection A. Once again, pick a $\varphi \in V_p$ and extend to $\tilde{\varphi} \in \Gamma(V)$; fix f with $f(p) = 0$ and $(\mathrm{d}f)(p) = \xi$ for a $\xi \in T_p^*V$. Then

$$\begin{aligned} D_A(f \cdot \tilde{\varphi})(p) &= \gamma_{\text{eval}} \circ g(\nabla^A(f\tilde{\varphi})(p)) = \gamma_{\text{eval}} \circ g((\mathrm{d}f \otimes \tilde{\varphi} + f\nabla^A\tilde{\varphi})(p)) \\ &= \gamma(\xi^*) \cdot \varphi \end{aligned}$$

where ξ^* is dual to ξ under the identification $T_p^*M \cong T_pM$ induced by g. Hence, $\sigma(D_A)(\xi) = \gamma(\xi^*)$ and D_A is elliptic (since Clifford multiplication with a fixed element is an isomorphism). Since D_A is formally self-adjoint, one typically finds $\operatorname{ind} D_A = 0$. However, there are ways to "break the symmetry" and obtain something interesting.

We now specialize to a closed, oriented, smooth (COS) 4-manifold M with Spin structure, $V = V_+ \oplus V_-$ and $D_A^+ : \Gamma(V_+) \to \Gamma(V_-)$, which is elliptic but not self-adjoint. A celebrated theorem then relates the Fredholm index of D_A^+ to a topological quantity:

Theorem 4.4 (Atiyah-Singer Index Theorem)**.**

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = \langle \hat{A}(M), [M] \rangle$$

where $\hat{A}(M) = 1 - (1/24)p_1(M) + \ldots$. On a 4-manifold, the degree 4 part is relevant. We obtain

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = -\frac{1}{24}\langle p_1(TM), [M] \rangle = -\frac{1}{8}\sigma(M) \tag{4.1}$$

Corollary 4.5. *If M is* Spin*, $\sigma(M)$ is divisible by 8.*

In fact, one can do better than this immediate corollary:

Theorem 4.6 (Rohlin)**.** *If M is a COS,* Spin *manifold of dimension four, then $\sigma(M) \equiv 0 \mod 16$.*

Proof. We use the Atiyah-Singer index formula: The kernel and cokernel of D_A^+ are $\mathbb{C}$-vector spaces with $\mathrm{ind}_{\mathbb{C}} D_A^+ = \dim_{\mathbb{C}} \ker D_A^+ - \dim_{\mathbb{C}} \operatorname{coker} D_A^+ = -\frac{1}{8}\sigma(M)$. In the case of a Spin structure, charge conjugation preserves the kernel and cokernel of D_A^+. Hence, these are in fact quaternionic vector spaces, hence their $\mathbb{C}$-dimensions are in fact even. Thus, the difference is even, i.e. $\sigma(M)/8 \equiv 0 \mod 2$. □

Remark 4.7. This shows that, in the intersection form, only even multiples of E_8 may occur. Hence, many simply connected 4-manifolds with even intersection form do not admit a smooth structure since if they did, they would be Spin.

For a Spin^c structure on a COS manifold M^4 which does not necessarily come from a Spin structure there is a generalization of the index formula:

$$\begin{aligned}
\mathrm{ind}_{\mathbb{C}} D_A^+ &= \left\langle \hat{A}(M) \cdot e^{c_1(L_{\mathfrak{s}})/2}, [M] \right\rangle \\
&= \left\langle \left(1 - \frac{1}{24} p_1(M) + \dots\right) \left(1 + \frac{1}{2} c_1(L_{\mathfrak{s}}) + \frac{1}{8} c_1^2(L_{\mathfrak{s}}) + \dots\right), [M] \right\rangle \\
&= \frac{1}{8} \left(\langle c_1^2(L_{\mathfrak{s}}), [M] \rangle - \sigma(M) \right)
\end{aligned}$$

4.2 The Weitzenböck Formula

Recall that in section 2.5.1, we defined the Dirac operator D on $\mathbb{R}^4$ such that $D^2 = \Delta$. We now wish to make a similar construction on closed 4-manifolds. Let X be a COS 4-manifold with a Spin^c-structure $\mathfrak{s}$, spinor bundle $V = V_+ \oplus V_-$ and characteristic line bundle $L_{\mathfrak{s}} = \det V_{\pm}$. Let $\hat{A}$ be a $U(1)$-connection on $L_{\mathfrak{s}}$. Together with the Levi-Cività connection of g, this yields a Spin^c-connection A on V (cf. corollary 2.66). Recall that in lemma 2.53, we obtained the following expression for the Dirac operator $D_A : \Gamma(V) \to \Gamma(V)$ with respect to a local orthonormal frame $\{e_1, \dots e_4\}$ of (TX, g):

$$D_A \phi = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}^A \phi$$

We now state the main result of the section.

Theorem 4.8 (Weitzenböck Formula)**.** *The Dirac operator satisfies*

$$D_A^2 = D_A \circ D_A = \nabla_A^* \nabla_A + \frac{1}{4} s_g + \frac{1}{2} \gamma \left(F_{\hat{A}} \right)$$

where $F_{\hat{A}} \in \Omega^2(X; \mathfrak{u}(1))$ is the curvature of $\hat{A}$, $\gamma\left(F_{\hat{A}}\right) \in \mathrm{End}(V)$ is the extension of Clifford multiplication to 2-forms and s_g is the scalar curvature of g which acts on V by multiplication.

We will prove this in small steps.

Definition 4.9 (Bochner Laplacian)**.** $\nabla_A^* \nabla_A : \Gamma(V) \to \Gamma(V)$ is called the Bochner Laplacian. Recall that the covariant derivative $\nabla_A : \Gamma(V) \to \Gamma(T^*X \otimes V)$, so we define its adjoint $\nabla_A^* : \Gamma(T^*X \otimes V) \to \Gamma(V)$ as its adjoint with respect to the L^2-inner product: $\langle \nabla_A^* \phi, \psi \rangle_{L^2} = \langle \phi, \nabla_A \psi \rangle_{L^2}$.

Lemma 4.10. *Let $\{e_i\}$ be a local orthonormal basis for X. Then for $\phi \in \Gamma(V)$, we have*

$$\nabla_A^* \nabla_A \phi = \sum_i \left(-\nabla_{e_i}^A \nabla_{e_i}^A \phi + \nabla_{\nabla_{e_i} e_i}^A \phi \right)$$

where ∇ is the Levi-Civita connection.

Proof. Suppose $\psi \in \Gamma(V)$ has compact support on the open set on which the local frame $\{e_i\}$ is defined and otherwise arbitrary. Then $\nabla_A^* \nabla_A \phi$ is characterized by its inner product with such ψ's. We have, after expanding $\nabla_A \phi$ in the given basis:

$$\langle \nabla_A^* \nabla_A \phi, \psi \rangle_{L^2} = \sum_i \int_X \langle \nabla_{e_i}^A \phi, \nabla_{e_i}^A \psi \rangle \mathrm{vol}_g$$

Here, the pointwise metric $\langle -, - \rangle$ is induced by g and the Hermitian metric on V. On the other hand, we may start on the other side of the identity we want to prove and use that ∇^A is compatible with the metric:

$$\int_X \left\langle \sum_i (-\nabla_{e_i}^A \nabla_{e_i}^A \phi + \nabla_{\nabla_{e_i} e_i}^A \phi, \psi \right\rangle \mathrm{vol}_g = \sum_i \int_X \big(- L_{e_i} \langle \nabla_{e_i}^A \phi, \psi \rangle) + \langle \nabla_{e_i}^A \phi, \nabla_{e_i}^A \psi \rangle + \langle \nabla_{\nabla_{e_i} e_i}^A \phi, \psi \rangle \big) \mathrm{vol}_g$$

Thus, all we need to show is that the first and last term cancel. Set $\eta(Y) = \langle \nabla_Y^A \phi, \psi \rangle$ to find:

$$\sum_i \int_X (-L_{e_i} \eta(e_i) + \eta(\nabla_{e_i} e_i)) \mathrm{vol}_g = -\sum_i \int_X (\nabla_{e_i} \eta)(e_i) \mathrm{vol}_g$$

Now, we expand in a local parallel frame $\{\omega^j\}$ to see that pointwise $\sum_{i,j} (\nabla_{e_i}(\eta_j \omega^j))(e_i) = \sum_i \partial_i \eta_i$. On the other hand, we have the following pointwise calculation:

$$\begin{aligned} *\mathrm{d} * \Big(\sum_j \eta_j \omega^j \Big) &= *\mathrm{d} \Big(\sum_j \eta_j (-1)^j \omega^0 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^3 \Big) = \Big(\sum_j \partial_j \eta_j \Big) * \mathrm{vol} \\ &= \sum_j \partial_j \eta_j \end{aligned}$$

On $\eta \in \Omega^1(X)$, we have $\mathrm{d}^* \eta \, \mathrm{vol}_g = - * \mathrm{d} \, * \eta \mathrm{vol}_g = \mathrm{d} * \eta$ (where we used that $\mathrm{d} * \eta \in \Omega^4(X)$ and hence $*(\mathrm{d} * \eta) \mathrm{vol}_g = \mathrm{d} * \eta$). Thus, we find

$$\sum_i \int_X (-L_{e_i} \eta(e_i) + \eta(\nabla_{e_i} e_i)) \mathrm{vol}_g = \int_X \mathrm{d}^* \eta \, \mathrm{vol}_g = \int_X \mathrm{d} * \eta = 0$$

by Stokes' theorem. This proves our assertion. □

We now make the following definition.

Definition 4.11 (Curvature). For A a Spin^c-connection on V, we define its curvature $F_A \in \Omega^2(X, \mathrm{End}(V))$ by

$$F_A(X, Y)\phi = \nabla_X^A \nabla_Y^A \phi - \nabla_Y^A \nabla_X^A \phi - \nabla_{[X,Y]}^A \phi$$

This is tensorial in X, Y, ϕ.

Thus, we have a map

$$\begin{array}{ccccc} \Gamma(\Lambda^2(T^*X) \otimes \mathrm{End}(V)) & \xrightarrow{\gamma \otimes \mathrm{Id}} & \Gamma(\mathrm{End}(V) \otimes \mathrm{End}(V)) & \xrightarrow{\text{comp.}} & \Gamma(\mathrm{End}(V)) \\ F_A & \longmapsto & & & \gamma(F_A) \end{array}$$

where comp. denotes composition of endomorphisms. If $\{e_i\}$ is a local orthonormal basis, with $\{\omega^i\}$ its dual basis, we have the following expressions:

$$F_A = \sum_{i,j} F_A(e_i, e_j)\omega^i \otimes \omega^j = 2\sum_{i<j} F_A(e_i, e_j)\omega^i \wedge \omega^j$$
$$\implies \gamma(F_A)(\phi) = 2\sum_{i<j} \gamma(e_i \wedge e_j) \circ F_A(e_i, e_j)(\phi) = \sum_{i,j} \gamma(e_i \wedge e_j)(F_A(e_i, e_j)(\phi))$$

Lemma 4.12. *The Dirac operator satisfies*

$$D_A^2 = \left(\nabla_A^* \nabla_A + \frac{1}{2}\gamma(F_A)\right) .$$

Note the appearance of F_A, not $F_{\hat{A}}$.

Proof. This is a tensorial equation hence we may use a local parallel frame $\{e_j\}$ in p. Then we simply compute:

$$D_A^2\phi = D_A\Big(\sum_i e_j \cdot \nabla_{e_j}^A \phi\Big) = \sum_{i,j} e_i \cdot \nabla_{e_i}^A (e_j \cdot \nabla_{e_j}^A \phi) = \sum_{i,j} e_i \cdot \big(e_j \cdot \nabla_{e_i}^A \nabla_{e_j}^A \phi\big)$$

Splitting this equation into terms $i = j$ and $i \neq j$ and using the defining properties of Clifford multiplication, we find:

$$D_A^2\phi = \sum_i -\nabla_{e_i}^A \nabla_{e_i}^A \phi + \sum e_i \cdot e_j \cdot (\nabla_{e_i}^A \nabla_{e_j}^A - \nabla_{e_j}^A \nabla_{e_i}^A)\phi$$

Observe that the last term is $\frac{1}{2}\gamma(e_i \wedge e_j)(\cdots)\phi$. Now use lemma 4.10 and our expression for the curvature (remembering that $\nabla e_i = 0$ and $[e_i, e_j] = 0$) and conclude

$$D_A^2\phi = \nabla_A^* \nabla_A \phi + \frac{1}{2}\gamma(F_A)(\phi)$$

□

The last question that needs to be settled is: How does F_A relate to $F_{\hat{A}}$? Without proof, we claim:

Lemma 4.13. *Locally, the following formula holds with respect to a local frame $\{e_i\}$:*

$$\gamma(F_A) = \gamma(F_{\hat{A}}) + \frac{1}{4}\sum_i \gamma(e_i)\gamma(R(A))$$

where

$$R(A) = \frac{1}{2}\sum_{j,k,l} R_{ijkl}\; e_j \wedge e_k \wedge e_l$$

and $R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$ is the Riemann curvature tensor.

Taking this at face value, we see that

$$\frac{1}{2}\gamma(F_A) = \frac{1}{2}\gamma(F_{\hat{A}}) + \frac{1}{8}\sum_{i,j,k,l} \gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)R_{ijkl}$$

and we want to show that this last term equals $\frac{1}{4}s_g$. Recall that $R_{iikl} = R_{ijkk} = 0$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (the first Bianchi identity). The first identity tells us we can simplify the second term to

$$\frac{1}{2}\sum_{\substack{i<j\\k<l}} \gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)R_{ijkl}$$

while the Bianchi identity shows that the terms where i, j, k or i, j, l are all pairwise different cancel out. Therefore, the only terms are contribute are those with $i = k$ and $j = l$ so we find that the second term equals

$$\frac{1}{2}\sum_{i<j}\gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_j)R_{ijij} = -\frac{1}{2}\sum_{i<j}R_{ijij} = \frac{1}{2}\sum_{i<j}R_{ijji}$$

where we used the definition of Clifford multiplication. Now recall that $\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y)$, i.e. $\mathrm{Ric}_{ij} = \sum_k g(R(e_k, e_i)e_j, e_k)$ and $s_g = \sum_i \mathrm{Ric}_{ii} = \sum_{i,j} R_{ijji} = 2\sum_{i<j} R_{ijji}$, we obtain:

$$\frac{1}{2}\gamma(F_A) = \frac{1}{2}\gamma(F_{\hat{A}}) + \frac{1}{4}s_g$$

This completes our proof of the Weitzenböck formula.

If M^4 is Spin and COS, the characteristic line bundle L is trivial, hence we may take $\hat{A} = \mathrm{d}$, hence $F_{\hat{A}} = 0$ so that

$$D_A^2 = \nabla_A^*\nabla_A + \frac{1}{4}s_g$$

By the Atiyah-Singer index theorem, $\mathrm{ind}_{\mathbb{C}}\, D_A^+ = -\frac{1}{8}\sigma(M)$. Since the Weitzenböck formula holds on $V_\pm$ with $D_A^2 = D_A^\mp D_A^\pm$, combining it with the Atiyah-Singer theorem yields the so-called "Lichnerowicz argument":

Theorem 4.14 (Lichnerowicz)**.** *If a COS, Spin 4-manifold admits a metric g such that $s_g > 0$, then $\sigma(X) = 0$.*

Proof. We will show that $\mathrm{ind}\, D_A^+ = 0$. First, observe that

$$\mathrm{ind}\, D_A^+ = \dim\ker D_A^+ - \dim\operatorname{coker} D_A^+ = \dim\ker D_A^+ - \dim\ker D_A^-$$

since $D_A^\mp$ is the formal adjoint of $D_A^\pm$. Thus, assume $\phi \in \ker D_A^\pm$. Then we find

$$0 = D_A^\pm D_A^\mp \phi = \nabla_A^*\nabla_A\phi + \frac{1}{4}s_g\phi$$

and taking the L^2-inner product with ϕ itself yields

$$\langle\nabla_A\phi, \nabla_A\phi\rangle_{L^2} + \frac{1}{4}\langle s_g\phi, \phi\rangle_{L^2} = \int_X \left(|\nabla^A\phi|^2 + \frac{1}{4}s_g|\phi|^2\right)\mathrm{vol}_g = 0$$

The terms must vanish individually, thus if $s_g > 0$ we see $\nabla^A\phi = 0$ and in fact $\phi = 0$, i.e. $\dim\ker D_A^\pm = 0$. □

Corollary 4.15. *If the manifold admits a metric with non-negative scalar curvature, $\phi \in \ker D_A^\pm$ still implies $\nabla^A\phi = 0$, i.e. ϕ must be parallel.*

Example 4.16.

(i) $S^2 \times S^2$ is Spin and admits a metric with positive scalar curvature, therefore $\sigma(S^2 \times S^2) = 0$.

(ii) $K3$ is Spin has $\sigma(K3) = -16$: This means it does not admit a metric with $s_g > 0$—but in fact $K3$, being a Calabi-Yau manifold, admits a Ricci-flat (hence $s_g \equiv 0$) Kähler metric.

(iii) The above theorem does not apply to manifolds of the form $\mathbb{CP}^2\#k\bar{\mathbb{CP}}^2$ ($k > 1$), which admit a metric with $s_g > 0$ and have nonzero signature, yet do not admit a Spin structure.

4.3 The Seiberg-Witten Equations

Let (X, g) be a COS Riemannian 4-manifold with a Spin^c-structure $\mathfrak{s}$, equipped with a Spin^c-connection A; let $\Gamma(V_+)$ denote the space of positive spinors, with $\Phi \in \Gamma(V_+)$ a positive spinor. Our main object of study in the following are the Seiberg-Witten (SW) equations for the pair (A, Φ).

The first SW equation, called the *Dirac equation* reads $D_A^+\Phi = 0$. For the second equation, called the *curvature equation*, we recall (cf. lemma 2.9) that γ induces an isomorphism $\bigwedge_+^2 T^*X \otimes \mathbb{C} \cong \mathrm{End}_0(V_+)$, where End_0 denotes the space of traceless endomorphisms. It maps real-valued self-dual forms to traceless, skew-Hermitian endomorphisms and imaginary-valued forms to traceless, Hermitian endomorphisms. In particular, $F_{\hat{A}}^+$ (where $\hat{A}$ is the $U(1)$-connection associated to A) corresponds, as an element of $\Omega_+^2(X, i\mathbb{R})$, to a traceless, Hermitian endomorphism under γ.

Now let $\Phi \in \Gamma(V_+)$ be a positive spinor. We define $\Phi \otimes \Phi^\dagger \in \mathrm{End}(V_+)$ by $(\Phi \otimes \Phi^\dagger)(\psi) = \Phi h(\Phi, \psi)$, where $h(-,-)$ is the Hermitian metric (anti-linear in the *first* entry) on $\Gamma(V_+)$. We denote its traceless part by $(\Phi \otimes \Phi^\dagger)_0$. Consider $\Phi = \binom{a}{b}$ in a basis for V_+. Then

$$\Phi \otimes \Phi^\dagger = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$$

so

$$(\Phi \otimes \Phi^\dagger)_0 = \begin{pmatrix} \frac{1}{2}\left(|a|^2 - |b|^2\right) & a\bar{b} \\ \bar{a}b & \frac{1}{2}\left(|b|^2 - |a|^2\right) \end{pmatrix}$$

is the desired trace-free endomorphism of V_+.

Definition 4.17. We define $\sigma(\Phi, \Phi) \in \Omega_+^2(X, i\mathbb{R})$ through the relation

$$\gamma(\sigma(\Phi, \Phi)) = (\Phi \otimes \Phi^\dagger)_0$$

The curvature equation reads $F_{\hat{A}}^+ = \sigma(\Phi, \Phi)$. Thus, the SW equations for (A, Φ) are:

$$D_A^+\Phi = 0$$
$$F_{\hat{A}}^+ = \sigma(\Phi, \Phi)$$

Since the physical interpretation of these equations is in terms of massless magnetic monopoles, they are sometimes called the *monopole equations*. Solutions (A, Φ) may also be called *monopoles*.

As mentioned in the introductory chapter, the monopole equations are nonlinear partial differential equations: The *Dirac equation* contains a term of second order in $A\Phi$, while the curvature equation is quadratic in Φ (through $\sigma(\Phi, \Phi)$). In applications, it is often necessary to *perturb* the curvature equation by a self-dual imaginary-valued form $\omega \in \Omega_+^2(X, i\mathbb{R})$. The ω-perturbed SW equations read:

$$D_A^+\Phi = 0$$
$$F_{\hat{A}}^+ = \sigma(\Phi, \Phi) + \omega$$

Definition 4.18 (SW Parameter Space)**.** The space of parameters for the SW equations on X is

$$\mathcal{P} = \{(g, \omega) \in \mathrm{Met}(X) \times \Omega_+^2(X, i\mathbb{R})\}$$

Definition 4.19 (SW Configuration Space)**.** The space

$$\mathcal{C}_{\mathfrak{s}} = \mathcal{A}_{\mathfrak{s}} \times \Gamma(V_+)$$

where $\mathcal{A}_{\mathfrak{s}}$ is the space of Spin^c-connections on V compatible with the Levi-Civita connection is called the Seiberg-Witten configuration space.

Note that by corollary 2.66, we can identify $\mathcal{A}_{\mathfrak{s}}$ with $\mathcal{A}(L_{\mathfrak{s}})$, the space of Hermitian connections on $L_{\mathfrak{s}}$, which is an affine space over the space $\Omega^1(X, i\mathbb{R})$ of imaginary-valued 1-forms over X.

Corollary 4.20. *The SW configuration space $\mathcal{C}_{\mathfrak{s}}$ is the product of a vector space and an infinite-dimensional affine space.*

Consider the map

$$
\begin{aligned}
f_\omega : \mathcal{C}_\mathfrak{s} &\longrightarrow i\Omega^2_+(X) \times \Gamma(V_-) \\
(A, \Phi) &\longmapsto (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi)
\end{aligned}
$$

then the *solution space* of the SW equations

$$\mathcal{Z}_\omega = \{(A, \Phi) \in \mathcal{C}_\mathfrak{s} |\ (A, \Phi) \text{ satisfy the SW equations}\} = f_\omega^{-1}(0)\ .$$

Definition 4.21. The *energy* of a pair $(A, \Phi) \in \mathcal{C}_\mathfrak{s}$ is given by

$$E(A, \Phi) = \int_X \left(|D^+_A \Phi|^2 + |F^+_{\hat{A}} - \sigma(\Phi, \Phi)|^2 + \frac{1}{8} s_g^2 \right) \mathrm{vol}_g - 4\pi^2 \langle c_1^2(L_\mathfrak{s}), [X] \rangle$$

The following proposition makes it manifest that this quantity is always positive.

Proposition 4.22. *We energy can be re-expressed as:*

$$E(A, \Phi) = \int_X \Big(|\nabla^A \Phi|^2 + \frac{1}{8} \big(s_g + |\Phi|^2 \big)^2 + |F^-_{\hat{A}}|^2 \Big) \mathrm{vol}_g$$

for all $(A, \Phi) \in \mathcal{C}_\mathfrak{s}$.

To prove this, we need some identities:

Lemma 4.23. *Equip* $\mathrm{End}(V_+)$ *with the inner product* $\langle A, B \rangle = \mathrm{tr}(AB^\dagger)$. *Then, for every* $\omega, \eta \in i\Lambda^2_+ T^* X$ *and* $\Phi \in V_+$, *we have:*

(i) $\langle \gamma(\omega), \gamma(\eta) \rangle = 4 \langle \omega, \eta \rangle$.

(ii) $\langle \gamma(\omega)\Phi, \Phi \rangle = 4 \langle \omega, \sigma(\Phi, \Phi) \rangle$.

(iii) $|\Phi|^4 = 8 \langle \sigma(\Phi, \Phi), \sigma(\Phi, \Phi) \rangle$.

Proof. This is proven in exercise 8.1. □

Proof of Proposition. Recall the Weitzenböck formula, which implies

$$\int_X |D^+_A \Phi|^2 \mathrm{vol}_g = \int_X \langle D^-_A D^+_A \Phi, \Phi \rangle \mathrm{vol}_g = \int_X \Big(|\nabla^A \Phi|^2 + \frac{1}{4} s_g |\Phi|^2 + \frac{1}{2} \langle \gamma(F^+_{\hat{A}})\Phi, \Phi \rangle \Big) \mathrm{vol}_g$$

Using the lemma, we have $\frac{1}{2} \langle \gamma(F^+_{\hat{A}} \Phi, \Phi \rangle = 2 \langle F^+_{\hat{A}}, \sigma(\Phi, \Phi) \rangle$. This term is canceled by the second term from:

$$\int_X |F^+_{\hat{A}} - \sigma(\Phi, \Phi)|^2 \mathrm{vol}_g = \int_X \big(|F^+_{\hat{A}}|^2 - 2 \langle F^+_{\hat{A}}, \sigma(\Phi, \Phi) \rangle + |\sigma(\Phi, \Phi)|^2 \big) \mathrm{vol}_g$$

Moreover, $|\sigma(\Phi, \Phi)|^2 = \frac{1}{8} |\Phi|^4$. Putting this term together with the term $\frac{1}{8} s_g^2$ and the scalar curvature term from $|D^+_A \Phi|^2$, we obtain

$$\frac{1}{8} \int_X (s_g + |\Phi|^2)^2 \mathrm{vol}_g$$

Now, we have arrived at

$$|D^+_A \Phi|^2 + |F^+_{\hat{A}} - \sigma(\Phi, \Phi)|^2 + \frac{1}{8} s_g^2 = |\nabla^A \Phi|^2 + \frac{1}{8} \big(s_g + |\Phi|^2 \big)^2 + |F^+_{\hat{A}}|^2$$

Thus, all that is left is to show that

$$-4\pi^2 \langle c_1^2(L_\mathfrak{s}), [X] \rangle = \int_X \big(|F^-_{\hat{A}}|^2 - |F^+_{\hat{A}}|^2 \big) \mathrm{vol}_g$$

This follows from *Chern-Weil theory*, which yields a definition of Chern classes in terms of curvature quantities. In particular, we have $c_1(L_\mathfrak{s}) = \frac{i}{2\pi}[F_{\hat{A}}]$. Thus, we find

$$-4\pi^2\langle c_1^2(L_\mathfrak{s}), [X]\rangle = \int_X F_{\hat{A}} \wedge F_{\hat{A}} = \int_X F_{\hat{A}}^+ \wedge F_{\hat{A}}^+ + F_{\hat{A}}^- \wedge F_{\hat{A}}^- = \int_X F_{\hat{A}}^+ \wedge (*F_{\hat{A}}^+) - F_{\hat{A}}^- \wedge (*F_{\hat{A}}^-)$$

Now, recall that the extension of the Hodge star operator to $\mathbb{C}$-valued forms is

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_g$$

Since $F_{\hat{A}}^\pm$ takes values in $i\mathbb{R}$, $\bar{F}_{\hat{A}}^\pm = -F_{\hat{A}}^\pm$ and we conclude:

$$-4\pi^2\langle c_1^2(L_\mathfrak{s}), [X]\rangle = \int_X \big(-|F_{\hat{A}}^+|^2 + |F_{\hat{A}}^-|^2\big)\mathrm{vol}_g$$

This completes the proof. ☐

Corollary 4.24. *If there is a solution to the SW equations, then*

$$\langle c_1^2(L_\mathfrak{s}), [X]\rangle \leq \frac{1}{32\pi^2}\int_X s_g^2 \mathrm{vol}_g$$

If equality holds, then every solution (A, Φ) has $E(A, \Phi) = 0$, and thus $\nabla^A\Phi = F_{\hat{A}}^- = s_g + |\Phi|^2 = 0$.

Proof. The previous proposition showed that $E(A, \Phi) \geq 0$, with the conditions of equality given by the second assertion.. If (A, Φ) solve the SW equations, our first expression for the energy becomes

$$E(A, \Phi) = \frac{1}{8}\int_X s_g^2 \mathrm{vol}_g - 4\pi^2\langle c_1^2(L_\mathfrak{s}), [X]\rangle \geq 0$$

and rearranging gives the desired inequality. ☐

4.4 Symmetries of the Seiberg-Witten Equations

4.4.1 Charge Conjugation

Recall the charge conjugation map $J : \bar{\mathfrak{s}} \to \mathfrak{s}$ from section 2.2.3. It induces a map $\tau : \mathcal{C}_{\bar{\mathfrak{s}}} \to \mathcal{C}_\mathfrak{s}$. Let A be a Spin^c-connection on $\bar{V}$: It induces a Spin^c-connection A^* on V, defined by $\nabla_X^{A^*}(J\Phi) = J\nabla_X^A\Phi$ for every $\Phi \in \Gamma\left(\bar{V}\right)$. Now we define τ by

$$\begin{aligned} \tau : \mathcal{C}_{\bar{\mathfrak{s}}} &\longrightarrow \mathcal{C}_\mathfrak{s} \\ (A, \Phi) &\longmapsto (A^*, J\Phi) \\ (\tilde{A}^*, J^{-1}\Psi) &\longleftarrow\!\shortmid (\tilde{A}, \Psi) \end{aligned}$$

We see that $\tau^2 = \mathrm{Id}$, i.e. τ is an involution.

Lemma 4.25. *A pair $(A, \Phi) \in \mathcal{C}_{\bar{\mathfrak{s}}}$ satisfies the SW equations for parameters (g, ω) if and only if $\tau(A, \Phi) \in \mathcal{C}_\mathfrak{s}$ satisfies them for $(g, -\omega)$.*

Proof. We first check the Dirac equation, combined with the fact that J commutes with γ:

$$D_{A^*}^+ J\Phi = \sum_i e_i \cdot \nabla_{e_i}^{A^*} J\Phi = \sum_i e_i \cdot J\nabla_{e_i}^A \Phi = J\sum_i e_i \cdot \nabla_{e_i}^A\Phi = JD_A^+\Phi$$

Thus, (A, Φ) satisfies the Dirac equation if and only if $(A^*, J\Phi)$ does. The curvature satisfies $F_{\hat{A}^*} = -F_{\hat{A}}$ since $\bar{V} \cong V^*$ and the curvature of the dual connection on the dual bundle is negative the original curvature. Furthermore, exercise 8.2 shows that $\sigma(J\Phi, J\Phi) = -\sigma(\Phi, \Phi)$. Thus, the curvature equation is satisfied by $(A^*, J\Phi)$ if we map ω to $-\omega$. ☐

4.4.2 The Action of the Gauge Group

The *gauge group* of the SW equations, $\mathcal{G} = C^\infty(X, S^1)$, has actions defined as follows.

- On $\mathcal{C}_\mathfrak{s}$ we have, for $u \in \mathcal{G}$,
$$(A, \Phi) \mapsto (A, \Phi) \cdot u := ((u^{-1})^* A, u\Phi) \,,$$
and the Spin^c-connection transforms as $\nabla^{(u^{-1})^* A} := u\nabla_A u^{-1}$.
- On $i\Omega^2_+(X) \times \Gamma(V_-) \supseteq f_\omega(\mathcal{C}_\mathfrak{s})$, we define an action by $(\eta, \psi) \mapsto (\eta, \psi) \cdot u := (\eta, u\psi)$.

Lemma 4.26. *f_ω is equivariant with respect to these actions of $\mathcal{G}$, i.e. $f_\omega((A, \Phi) \cdot v) = f_\omega(A, \Phi) \cdot v$.*

Proof. We can simply write out
$$f_\omega((A, \Phi) \cdot v) = f_\omega((v^{-1})^* A, v\Phi) = (F^+_{\widehat{(v^{-1})^* A}} - \sigma(v\Phi, v\Phi) - \omega, D^+_{(v^{-1})^* A}(v\Phi))$$
Firstly, $\sigma(v\Phi, v\Phi) = \sigma(\Phi, \Phi)$, since by definition of σ, we have
$$\gamma(\sigma(\Phi, \Phi)) = (\Phi \otimes \Phi^\dagger)_0 = (v\Phi \otimes \bar{v}\Phi^\dagger)_0 = \gamma(\sigma(v\Phi, v\Phi))$$
where we used that v takes values in S^1, i.e. $\bar{v} = v^{-1}$. Next, we consider the curvature, using $\nabla^{\widehat{(v^{-1})^* A}}\Phi = v \cdot \nabla^A(v \cdot \Phi)$ where $\cdot$ denotes the corresponding group action (which we need not explicitly know) and will be omitted in the following:
$$\begin{aligned} F^+_{\widehat{(v^{-1})^* A}}(X, Y)s &= \big([\nabla_X^{\widehat{(v^{-1})^* A}}, \nabla_Y^{\widehat{(v^{-1})^* A}}] - \nabla_{[X,Y]}^{\widehat{(v^{-1})^* A}}\big)s \\ &= v[\nabla_X, \nabla_Y]v^{-1}s - v\nabla_{[X,Y]}v^{-1}s \\ &= v\nabla_X((L_Y v^{-1})s + v^{-1}\nabla_Y s + \dots \\ &= v(L_X L_Y v^{-1})s + v(L_Y v^{-1})\nabla_X s + v(L_X v^{-1})\nabla_Y s + \nabla_X \nabla_Y s + \dots \end{aligned}$$
Note that the middle terms are symmetric in X, Y, hence will disappear. The first term will cancel against one term of $v(L_{[X,Y]}v^{-1})$, so only the last term remains. This shows that we recover $F^+_{\hat{A}}(X, Y)s$.

Finally, we check that $D^+_{(v^{-1})^* A} v\Phi = vD^+_A\Phi$. But this is immediate:
$$D^+_{(v^{-1})^* A} v\Phi = \sum_i e_i \cdot \nabla_{e_i}^{(v^{-1})^* A} v\Phi = \sum_i e_i \cdot v\nabla^A_{e_i}\Phi = vD^+_A\Phi$$
completing our proof that $f_\omega((A, \Phi) \cdot v) = (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, vD^+_A\Phi) = f_\omega(A, \Phi) \cdot v$. □

Lemma 4.27. *If X^4 is connected, the stabilizer, $\mathcal{G}_{(A,\Phi)}$ of $(A, \Phi) \in \mathcal{C}_\mathfrak{s}$ is given by*
$$\mathcal{G}_{(A,\Phi)} = \begin{cases} \{1\} & \textit{if } \Phi \not\equiv 0 \,, \\ \mathrm{U}(1) & \textit{if } \Phi \equiv 0 \,. \end{cases}$$

Proof. $u \in \mathcal{G}_{(A,\Phi)}$ means precisely $(A, \Phi) \cdot u = (A, \Phi)$, i.e. $u\nabla^A u^{-1} = \nabla^A$ and $u\Phi = \Phi$. Since $\mathrm{Ad}(u^{-1})$ is trivial, we have $u\nabla^A u^{-1} = \nabla^A + u\mathrm{d}(u^{-1})$. This equals ∇^A if and only if $u\mathrm{d}(u^{-1}) = -u^{-1}\mathrm{d}u = 0$, hence $u \in S^1$ is constant. If $\Phi \equiv 0$, and $u \in S^1$ is an element of the stabilizer. But if Φ does not identically vanish, $v \equiv 1$ is the only element in the stabilizer. □

Correspondingly, we make the following definition:

Definition 4.28. The space of *irreducible configurations* is
$$\mathcal{C}^*_\mathfrak{s} = \{(A, \Phi) \in \mathcal{C}_\mathfrak{s} \mid \Phi \not\equiv 0\}$$
$\mathcal{C}_\mathfrak{s} \setminus \mathcal{C}^*_\mathfrak{s}$ is the set of *reducible* solutions.

5 Topology of the Configuration Space

So far we have worked with smooth manifolds. To define the SW invariants, we have to introduce additional structures on the (infinite-dimensional!) configuration space $\mathcal{C}_{\mathfrak{s}}$. In particular, we need to topologize it. The best functional framework for such purposes is supplied by the machinery of Sobolev spaces.

Consider a vector bundle $E \to X$ with a Hermitian metric. On smooth sections $\Gamma(E)$, define a norm

$$\|s\|_k^p := \left(\int_X \left(|s|^p + |\nabla s|^p + \ldots + |\nabla^k s|^p \right) \mathrm{vol}_g \right)^{1/p}$$

for $p,\ k \in \mathbb{N}$.

Definition 5.1 (Sobolev Space)**.** The Banach space completion of $\Gamma(E)$ with respect to $\|\cdot\|_k^p$ is a Sobolev space of E, denoted $L_k^p(E)$. We write $L^p(E)$ for $L_0^p(E)$.

In our discussion, we will need the following assumptions on the quantities that appear in our study of the SW equations:[8]:

- Positive spinors $\Phi \in \Gamma(V_+)$ lie in $L_5^2(V_+)$.
- Sections $i\Lambda_+^2(X) \times V_-$ are elements of $L_4^2(i\Lambda_+^2(X) \times V_-)$.
- $\mathcal{A}_{\mathfrak{s}} \in L_5^2(\mathcal{A})$, i.e. of the form $\hat{A}_0 + a$ for $\hat{A}_0$ a smooth connection on $L_{\mathfrak{s}}$ and $a \in iL_5^2(T^*X)$.
- $\mathcal{G}$ consists of maps in $L_6^2(X, S^1)$.

We will slightly abuse notation and keeping using the old symbols when we are in actuality referring to corresponding Sobolev spaces.

Lemma 5.2. *The L_6^2-gauge group $\mathcal{G}$ is an infinite-dimensional Abelian Hilbert Lie group*[9]*. It acts smoothly on the L_5^2-configuration space $\mathcal{C}_{\mathfrak{s}}$, and on L_4^2-sections of $i\Lambda_+^2(X) \times V_-$.*

Proof. $\mathcal{G}$ is a Hilbert manifold; we just need to check that the multiplication makes sense. For that, we view $\mathcal{G}$ as a subset of $L_6^2(X;\mathbb{C})$ and use a *Sobolev multiplication theorem* to conclude that multiplication $L_6^2(X,\mathbb{C}) \times L_6^2(X,\mathbb{C}) \to L_6^2(X,\mathbb{C})$ is a bounded map, hence $\mathcal{G}$ has an Abelian Lie group structure. $\square$

Definition 5.3 (SW Base and Moduli Spaces)**.** We call $\mathcal{B} := \mathcal{C}_{\mathfrak{s}}/\mathcal{G}$ the base space of the SW equations. The moduli space is $\mathcal{M}_\omega := \mathcal{Z}_\omega/\mathcal{G} \subset \mathcal{B}$.

5.1 The Linearized Seiberg-Witten Equations

5.1.1 The Elliptic Complex

It is a fact, which we will not prove, that $f_\omega : \mathcal{C}_{\mathfrak{s}} \to i\Omega_+^2 \times \Gamma(V_-)$ is a smooth map. Therefore, its differential gives the *linearization* of the SW equations.

Lemma 5.4. *For $\omega \in iL_4^2(\Lambda_+^2 T^*X)$ and (A, Φ) in the L_5^2-configuration space $\mathcal{C}_{\mathfrak{s}}$, the differential of f_ω is given by:*

$$\begin{aligned} \mathcal{T}_{(A,\Phi)} f_\omega : i\Omega^1(X) \times \Gamma(V_+) &\longrightarrow i\Omega_+^2(X) \times \Gamma(V_-) \\ (a, \varphi) &\longmapsto (2\mathrm{d}^+ a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi), D_A^+ \varphi + \gamma(a)\Phi) \end{aligned}$$

[8]The last two items in this list are not covered by our definition, since these spaces are not those of sections of a vector bundle. However, there are ways to extend the Sobolev space construction to these spaces also: in the case of $\mathcal{G}$, for instance, one can obtain the Sobolev space of sections of the line bundle $\mathbb{C} \to X$ and then just restrict to sections with unit length.

[9]See e.g. here for some more information.

Proof. The domain and codomain are correct, since $\mathcal{A}_{\mathfrak{s}}$ is an affine space over $i\Omega^1(X)$ while $\Gamma(V_+)$ is a vector space, as is the target space. Now consider a curve $(A+ta, \Phi)$ through (A, Φ). Then we find

$$\mathcal{T}_{(A,\Phi)} f_\omega(a, 0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f_\omega(A+ta, \Phi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (F^+_{\widehat{A+ta}} - \sigma(\Phi, \Phi) - \omega, D^+_{A+ta}\Phi)$$

It is a fact which we will not explain that $F^+_{\widehat{A+ta}} = F^+_{\hat{A}} + 2t\mathrm{d}^+ a$ (the factor 2 is surprising: It arises because of some technicalities). Furthermore,

$$D^+_{A+ta}\Phi = \sum_i e_i \cdot \left(\nabla^A_{e_i} + ta(e_i)\right)\Phi = D^+_A\Phi + t\gamma(a)\Phi$$

Thus we are led to conclude that

$$\mathcal{T}_{(A,\Phi)} f_\omega(a, 0) = (2\mathrm{d}^+ a, \gamma(a)\Phi)$$

Proceeding similarly it is not hard to show

$$\mathcal{T}_{(A,\Phi)} f_\omega(0, \varphi) = (-\sigma(\Phi, \varphi) - \sigma(\varphi, \Phi), D^+_A\varphi)$$

Putting these results together yields the required result. $\square$

Let us now examine the "infinitesimal" (or linearized) action induced by $\mathcal{C}_{\mathfrak{s}} \curvearrowleft \mathcal{G}$.

Lemma 5.5. *Fix $(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}$. The action $\mathcal{C}_{\mathfrak{s}} \curvearrowleft \mathcal{G}$ induces a map*

$$\begin{aligned} \mathfrak{g} = i\Omega^0(X) &\xrightarrow{L_{(A,\Phi)}} T_{(A,\Phi)}\mathcal{C}_{\mathfrak{s}} = i\Omega^1(X) \times \Gamma(V_+) \\ \xi &\longmapsto (-\mathrm{d}\xi, \xi\Phi) \end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$, and $T_{(A,\Phi)}\mathcal{C}_{\mathfrak{s}}$ is the tangent space of $\mathcal{C}_{\mathfrak{s}}$ at (A, Φ).

Proof. We use the fact that $\exp(t\xi)$ has tangent vector ξ at $t = 0$ to compute:

$$\begin{aligned} L_{(A,\Phi)}\xi &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (A, \Phi) \cdot \exp(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (A + \exp(t\xi)\mathrm{d}(\exp(-t\xi)), \exp(t\xi)\Phi) \\ &= (-\mathrm{d}\xi, \xi\Phi) \end{aligned}$$

$\square$

This leads into the following result.

Proposition 5.6. *For fixed (A, Φ), we consider the composition*

$$i\Omega^0(X) \xrightarrow{L_{(A,\Phi)}} i\Omega^1 \times \Gamma(V_+) \xrightarrow{\mathcal{T}_{(A,\Phi)} f_\omega} i\Omega^2_+(X) \times \Gamma(V_-)$$

If $D^+_A\Phi = 0$, then for all $\omega \in i\Omega^2_+(X)$, the above is an elliptic complex with index (i.e. Euler characteristic[10]) $-\frac{1}{4}\left(c_1^2(L_{\mathfrak{s}}) - (2\chi(X) + 3\sigma(X))\right)$.

Proof. To show that we are dealing with a complex, we must show $\mathcal{T}_{(A,\Phi)} f_\omega \circ L_{(A,\Phi} = 0$. Using our previous lemmata, we find for $\xi \in i\Omega^0(X)$:

$$\mathcal{T}_{(A,\Phi)} f_\omega \circ L_{(A,\Phi)}(\xi) = \mathcal{T}_{(A,\Phi)} f_\omega(-\mathrm{d}\xi, \xi\Phi) = (-2\mathrm{d}^+\mathrm{d}\xi - \sigma(\Phi, \xi\Phi) - \sigma(\xi\Phi, \Phi), D^+_A(\xi\Phi) - \gamma(\mathrm{d}\xi)\Phi)$$

Clearly $\mathrm{d}^+\mathrm{d}\xi = 0$ while

$$\sigma(\Phi, \xi\Phi) + \sigma(\xi\Phi, \Phi) = \gamma^{-1}((\Phi \otimes (\xi\Phi)^\dagger)_0) + \gamma^{-1}(((\xi\Phi) \otimes \Phi^\dagger)_0) = -\xi\sigma(\Phi, \Phi) + \xi\sigma(\Phi, \Phi)$$

[10] As usual, the Euler characteristic is defined to be the alternating sum of the dimensions of cohomology vector spaces.

since $\bar{\xi} = -\xi$. Moreover,

$$D_A^+(\xi\Phi) = \sum_i e_i \cdot ((L_{e_i}\xi)\Phi + \xi\nabla_{e_i}\Phi) = \sum_i (e_i \cdot L_{e_i}\xi)\Phi + \xi D_A^+\Phi = \gamma(\mathrm{d}\xi)\Phi + \xi D_A^+\Phi$$

Thus, if $D_A^+\Phi = 0$ we have a complex, since the first term cancels.

To compute the index, we need the symbol of this differential operator. Recall from the discussion of the symbol (cf. section 4.1) that we pick a smooth function which vanishes at a given point. In this case, that means that we only need to take into account the terms which feature derivatives.

In the case at hand, the highest order terms are:

$$\begin{array}{ccccc} i\Omega^0(X) & \longrightarrow & i\Omega^1(X) \times \Gamma(V_+) & \longrightarrow & i\Omega^2_+ \times \Gamma(V_-) \\ \xi & \longmapsto & (-\mathrm{d}\xi, 0) & & \\ & & (a, \varphi) & \longmapsto & (2\mathrm{d}^+ a, D_A^+\varphi) \end{array}$$

This leads us to consider the separate (decoupled) complexes

$$i\Omega^0(X) \xrightarrow{-\mathrm{d}} i\Omega^1(X) \xrightarrow{\mathrm{d}^+} i\Omega^2_+(X)$$

$$0 \longrightarrow \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-)$$

These have finite dimensional cohomology groups and the direct sum of them yields the cohomology groups of the total complex. Observe that the former is simply the the half-de Rham complex we discussed before: It has index $\frac{1}{2}(\chi(X) + \sigma(X))$. The latter, on the other hand, has index simply given by $-\operatorname{ind}_{\mathbb{R}} D_A^+ = -2\operatorname{ind}_{\mathbb{C}} D_A^+$ (since D_A^+ is the *second* map). The Atiyah-Singer index theorem tells us that $\operatorname{ind}_{\mathbb{C}} D_A^+ = \frac{1}{8}(c_1^2(L_{\mathfrak{s}}) - \sigma(X))$. Putting our results together, we find total (real) index

$$\frac{1}{2}(\chi(X) + \sigma(X)) - \frac{1}{4}(c_1^2(L_{\mathfrak{s}}) - \sigma(X)) = \frac{1}{4}(2\chi(X) + 3\sigma(X) - c_1^2(L_{\mathfrak{s}}))$$

as claimed. □

Remark 5.7. It is a general that

$$\frac{1}{4}\left(c_1^2(L_{\mathfrak{s}}) - (2\chi(X) + 3\sigma(X))\right) = c_2(V_+)$$

This follows from

$$p_1(\Lambda^2_+(X)) = 2\chi(X) + 3\sigma(X) = c_1^2(V_+) - 4c_2(V_+) = c_1^2(L_{\mathfrak{s}}) - 4c_2(V_+)$$

This is discussed in exercise 8.3.

5.1.2 Implicit Function Theorem for Banach Manifolds

Here, we proceed without proofs, simply quoting some facts from the theory of Banach manifolds. Let $f : X \to Y$ be a smooth Fredholm map between Banach manifolds, i.e. $\mathcal{T}_x f$ is Fredholm for all $x \in X$. Now let $x_0 \in X$, $y_0 = f(x_0) \in Y$. We also define $K = \ker \mathcal{T}_{x_0} f$, and $C = \operatorname{coker} \mathcal{T}_{x_0} f$. Then there exist local charts (U, κ) and (V, κ') around x_0 and y_0 such that

$$\kappa : U \to B \oplus K, \text{mapping } x_0 \mapsto 0$$
$$\kappa' : V \to B \oplus C, \text{mapping } y_0 \mapsto 0$$

where B is the model Banach space for X and Y. In these charts, $F = \kappa' \circ f \circ \kappa^{-1} : B \oplus K \to B \oplus C$ is given on an open set $W \subset B \oplus K$ by $F(b, k) = (b, \psi(b, k)) \in V \oplus C$, where $\psi : W \to C$, $(b, k) \mapsto \psi(b, k) =: \psi_b(k)$ is a smooth map. Then we have $f^{-1}(y_0) \cong F^{-1}(0,0) \cong \psi_0^{-1}(0)$. The map $\psi_0 : W' \subset K \to C$ is an example of a *Kuranishi map*. If $C = 0$, then $f^{-1}(y_0)$ is homeomorphic to an open neighborhood of 0 in K. The conclusion is an analog of the implicit function theorem, i.e. in a neighborhood of x_0, $f^{-1}(y_0)$ is a smooth manifold of dimension equal to the dimension of K.

Definition 5.8. For $(A, \Phi) \in \mathcal{Z}_\omega$, let $H^i_{(A,\Phi)}$ be the ith cohomology group of the elliptic complex (from proposition 5.6). Its index is $\dim H^0_{(A,\Phi)} - \dim H^1_{(A,\Phi)} + \dim H^2_{(A,\Phi)}$.

Our discussion in the previous section shows that the index is independent of the choice of (A, Φ), though the individual cohomology groups may well depend on this choice.

Lemma 5.9. *Let X be CCOS. Then*

$$H^0_{(A,\Phi)} \cong \mathfrak{g} = \begin{cases} 0 & \text{if } \Phi \not\equiv 0 \\ \mathbb{R} & \text{if } \Phi \equiv 0 \end{cases}$$

Proof. $\xi \in H^0_{(A,\Phi)} \Leftrightarrow L_{(A,\Phi)}\xi = (0,0)$. This means that $\mathrm{d}\xi = 0$ and $\xi\Phi = 0$. Thus, ξ must be (locally, hence globally) constant and if $\Phi \equiv 0$ there is no further condition. If $\Phi \not\equiv 0$, we see $H^0_{(A,\Phi)} = 0$. □

Let us assume for now that (A, Φ) is an irreducible solution of the SW equations, i.e. $H^0_{(A,\Phi)} = 0$. Let S be a local slice for the $\mathcal{G}$-action on $\mathcal{C}_\mathfrak{s}$ around (A, Φ): This means that a neighborhood of (A, Φ) admits a smoothly embedded (closed) Hilbert submanifold S and the neighborhood is diffeomorphic to $S \times \mathcal{G}$ (i.e. we can picture S as a transverse submanifold to the $\mathcal{G}$-orbits).

Consider $f_\omega|_S : S \to i\Omega^2_+(X) \times \Gamma(V_-)$. This is a Fredholm map between Banach spaces, and a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is equal to $(f_\omega|_S)^{-1}(0)$. There also exists a Kuranishi map $\psi : H^1_{(A,\Phi)} \to H^2_{(A,\Phi)}$, which makes sense since

$$\begin{aligned} H^1_{(A,\Phi)} &= \ker \mathcal{T}_{(A,\Phi)} f_\omega / \operatorname{im} L_{(A,\Phi)} = \ker \mathcal{T}_{(A,\Phi)}\big(f_\omega|_S\big) \\ H^2_{(A,\Phi)} &= \operatorname{coker} \mathcal{T}_{(A,\Phi)} f_\omega = \operatorname{coker} \mathcal{T}_{(A,\Phi)}\big(f_\omega|_S\big) \end{aligned}$$

Now assume $H^2_{(A,\Phi)} = 0$ as well—transversality of f_ω to $(0,0)$ (to be established in theorem 5.38) ensures that this holds for a neighborhood of $(A, \Phi) \in \mathcal{Z}_\omega$. Then an open neighborhood of $[A, \Phi]$ in $\mathcal{M}_\omega$ looks like an open neighborhood of 0 in $H^1_{(A,\Phi)}$; in particular, around $[A, \Phi]$, $\mathcal{M}_\omega$ is a smooth manifold with dimension equal to the dimension of $H^1_{(A,\Phi)}$. Our discussion of the index of the elliptic complex shows:

Proposition 5.10. *If $H^0_{(A,\Phi)} = 0 = H^2_{(A,\Phi)}$, a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is a smooth manifold of dimension*

$$\dim^{\exp}\mathcal{M}_\omega := \frac{1}{4}\big(c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X))\big)$$

We call $\dim^{\exp}\mathcal{M}_\omega$ the "expected dimension" of the moduli space. □

In the reducible case, the analogous result is:

Proposition 5.11. *If $H^0_{(A,\Phi)} \cong \mathbb{R}$, $H^2_{(A,\Phi)} = 0$, then a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is the quotient of a smooth manifold of dimension $\dim^{\exp}\mathcal{M}_\omega + 1$ by a $\mathrm{U}(1)$-action.*

Proof. We still have the Kuranishi map $\psi : H^1_{(A,\Phi)} \to 0$ and the constant gauge transformations (i.e. elements of $\mathcal{G}_{(A,\Phi)}$) act on $\mathcal{A}_\mathfrak{s} \times \Gamma(V_+)$ and $i\Omega^2_+(X) \times \Gamma(V_-)$. The actions descends to a $\mathrm{U}(1)$-action on $H^1_{(A,\Phi)}$ and $H^2_{(A,\Phi)}$. The index of the elliptic complex is now given by

$$\dim H^0_{(A,\Phi)} - \dim H^1_{(A,\Phi)} + \dim H^2_{(A,\Phi)} = 1 - \dim H^1_{(A,\Phi)}$$

Hence

$$\dim H^1_{(A,\Phi)} = 1 + \frac{1}{4}\big(c_1^2(L_{\mathfrak{s}}) - (2\chi(X) + 3\sigma(X))\big) = 1 + \dim^{\exp}\mathcal{M}_\omega$$

Thus, a neighborhood of $\mathcal{M}_\omega$ is diffeomorphic to a $\mathrm{U}(1)$-quotient of an open subset of $H^1_{(A,\Phi)}$, which has dimension $1 + \dim^{\exp}\mathcal{M}_\omega$. □

5.2 The Structure of the Gauge Group

Definition 5.12. We define the *degree* of a map

$$\begin{aligned} \deg : \mathcal{G} &\longrightarrow H^1(X;\mathbb{Z}) \\ u &\longmapsto u^*\mu \end{aligned}$$

where μ is a fixed generator of $H^1(S^1;\mathbb{Z})$.

Proposition 5.13. $[X, S^1] \cong H^1(X;\mathbb{Z})$

Proof. Homotopy classes of maps $X \to S^1$ are in bijection with maps $\pi_1(X) \to \mathbb{Z}$, but since $\mathbb{Z}$ is Abelian and $H_1(X)$ is the Abelianization of $\pi_1(X)$, they factor through $H_1(X)$, hence are in bijection with maps $H_1(X) \to \mathbb{Z}$. But those are precisely elements of $H^1(X;\mathbb{Z})$. □

Corollary 5.14. $\deg u = 0$ *if and only if* $u \simeq$ *const.*

Definition 5.15. We give the subgroup of null-homotopic elements of $\mathcal{G}$ a special name:

$$\mathcal{G}_0 = \ker \deg = \{u \in \mathcal{G} |\, \deg u = 0 \Leftrightarrow u \simeq \text{const.}\}$$

The following is an elementary result from covering theory:

Lemma 5.16. $\deg u = 0$ *if and only if* $u = e^{if}$ *with* $f : X \to \mathbb{R}$ *globally defined.*

Proof. $F : X \to S^1$ lifts to $\mathbb{R}$, i.e. $F = e^{if}$ for $f : X \to \mathbb{R}$, if and only if $F_*\pi_1(X)$ is trivial in $\pi_1(S^1) \cong H_1(S^1;\mathbb{Z})$, i.e. $\deg F = 0$. □

Corollary 5.17. $\mathcal{G}_0$ *is the connected component of* $1 \in \mathcal{G}$ *and* $\mathcal{G}/\mathcal{G}_0 \cong H^1(X;\mathbb{Z}) = \mathbb{Z}^{b_1(X)}$*, where we used that* $\operatorname{Tor} H^1(X;\mathbb{Z}) \cong \operatorname{Tor} H_0(X;\mathbb{Z}) = 0$.

Consider the following two subgroups of $\mathcal{G}_0$:

- $\mathrm{U}(1) = \{e^{ic} \mid c \in \mathbb{R}\}$.
- $\mathcal{G}^\perp := \{e^{if} \mid f \in L^2_6(X),\ \int_X f \mathrm{vol}_g = 0\}$.

Proposition 5.18. *There is an isomorphism*

$$\begin{aligned} \mathrm{U}(1) \times \mathcal{G}^\perp &\longrightarrow \mathcal{G}_0 \\ (e^{ic}, e^{if}) &\longmapsto e^{i(c+f)} \end{aligned}$$

Proof. Let $h = e^{if}$ and set $\lambda_h = \exp\Big(\frac{i}{\mathrm{vol}X}\int_X f\mathrm{vol}_g\Big)$. Note that this is well-defined, since if $e^{if} = e^{if'}$ then $f = f' + 2\pi i k$ for $k \in \mathbb{Z}$, hence $\int_X f'\mathrm{vol}_g = \int_X f\mathrm{vol}_g + 2\pi i k \mathrm{vol}X$. This allows us to give an inverse of the map: $\mathcal{G}_0 \to \mathrm{U}(1) \times \mathcal{G}^\perp$, given by $h \mapsto (\lambda_h, \lambda_h^{-1}h)$. Note that

$$\lambda_h^{-1}h = \exp\Big(i\Big(f - \frac{1}{\mathrm{vol}X}\int_X f\mathrm{vol}_g\Big)\Big) =: e^{if'}$$

and

$$\int_X f'\mathrm{vol}_g = \int_X f - \frac{1}{\mathrm{vol}X} \cdot \mathrm{vol}X \int_X f\mathrm{vol}_g = 0$$

as required. □

Definition 5.19. $u \in \mathcal{G}$ is called *harmonic* if $\alpha = u\mathrm{d}u^{-1}$ is harmonic. The set of harmonic maps will be denoted by $\mathcal{G}^h$.

Lemma 5.20. *$\mathcal{G}^h$ is a subgroup of $\mathcal{G}$.*

Proof. Let $u, v \in \mathcal{G}^h$. Then $(uv)\mathrm{d}((uv)^{-1}) = u\mathrm{d}u^{-1} + v\mathrm{d}v^{-1}$. Since uu^{-1} is constant, $u\mathrm{d}u^{-1} = -u^{-1}\mathrm{d}u$. Thus, $\mathcal{G}^h$ is closed under multiplication and inversion. □

Note that $0 = \mathrm{d}^2(uu^{-1}) = 2(\mathrm{d}u)(\mathrm{d}u^{-1}) = 2\mathrm{d}\alpha$. Hence, for harmonicity one needs only check $\mathrm{d}^*\alpha = 0$.

Proposition 5.21. *For an arbitrary $u \in \mathcal{G}$, there exists an (up to a constant) unique $f_u : X \to \mathbb{R}$ such that ue^{-if_u} is harmonic.*

In the proof, we will need to use the *Green's operator* for $\Delta : \Omega^k(X) \to \Omega^k(X)$. Let $H : \Omega^k(X) \to \mathcal{H}^k(X)$ be the orthogonal projection.

Theorem 5.22. *There exists a Green's operator G for Δ, given by $G : \Omega^k(X) \to (\mathcal{H}^k(X))^\perp \subset \Omega^k(X)$, which maps α to the unique $\omega \in (\mathcal{H}^k(X))^\perp$ such that $\Delta\omega = \alpha - H(\alpha)$.*

Remark 5.23. The Green's operator has the properties $H + \Delta G = \mathrm{Id} = H + G\Delta$, and $HG = GH$.

Proof of Proposition. Set $\beta = u\mathrm{d}u^{-1} \in \Omega^1(X)$ and define a function $f = iG(\mathrm{d}^*\beta)$. We claim that $H(\mathrm{d}^*\beta) = 0$. To see this, let $\mathbb{R} \ni c \in \mathcal{H}^0(X)$. Then clearly $\langle \mathrm{d}^*\beta, c\rangle_{L^2} = \langle \beta, \mathrm{d}c\rangle_{L^2} = 0$, i.e. $\mathrm{d}^*\beta \in (\mathcal{H}^0(X))^\perp$: β has no component in $\mathcal{H}^0(X)$.

Thus, $\Delta f = i(\Delta G + H)(\mathrm{d}^*\beta) = i\mathrm{d}^*\beta$. We claim that $v = ue^{-if}$ is harmonic, i.e. we have a harmonic form

$$v\mathrm{d}v^{-1} = u\mathrm{d}u^{-1} + \mathrm{d}e^{if} = u\mathrm{d}u^{-1} + i\mathrm{d}f = \beta + i\mathrm{d}f$$

As remarked before, all we need to do is check $\mathrm{d}^*(v\mathrm{d}v^{-1}) = 0$:

$$\mathrm{d}^*(v\mathrm{d}v^{-1}) = \mathrm{d}^*\beta + i\mathrm{d}^*\mathrm{d}f = \mathrm{d}^*\beta - \mathrm{d}^*\beta = 0$$

Indeed, $v\mathrm{d}v^{-1}$ is harmonic. Uniqueness up to constant follows from our definition of f. □

After modifying this function by an appropriate constant, we obtain:

Corollary 5.24. *Given an arbitrary gauge transformation $u : X \to S^1$, there exists a unique $f_u : X \to \mathbb{R}$ such that $\int_X f_u \mathrm{vol}_g = 0$ and ue^{-if} is harmonic.*

Corollary 5.25. *The map*

$$\begin{aligned} \mathcal{G}^\perp \times \mathcal{G}^h &\longrightarrow \mathcal{G} \\ (e^{if}, g) &\longmapsto e^{if}g \end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned} \mathcal{G} &\longrightarrow \mathcal{G}^\perp \times \mathcal{G}^h \\ u &\longmapsto (e^{if_u}, ue^{-if_u}) \end{aligned}$$

Recalling that $\mathcal{G}_0 = \mathrm{U}(1) \times \mathcal{G}^{\perp}$ and $\mathcal{G}/\mathcal{G}_0 \cong H^1(X;\mathbb{Z})$, we conclude $H^1(X;\mathbb{Z}) \cong \mathcal{G}/\mathcal{G}_0 \cong \mathcal{G}^h/\mathrm{U}(1)$. We obtain a short exact sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{G}^h \xrightarrow{\deg} H^1(X;\mathbb{Z}) \longrightarrow 0$$

of Abelian groups ($\mathbb{Z}$-modules). It ends in a vector space, i.e. a free module, and such a short exact sequence always splits. To show this, it suffices to construct a section of $\deg$, i.e. a map $v : H^1(X;\mathbb{Z}) \to \mathcal{G}^h$ such that $\deg \circ v = \mathrm{Id}$. The proof that this is always possible if the sequence ends in a vector space is not hard: Take a basis $\{e_j\}$ of the vector space and define $v(e_j)$ to be *any* lift of e_j: Extend by linearity. Adopting the notation $\mathcal{G}^h_v$ for the image of $H^1(X;\mathbb{Z})$ under v, we then obtain

$$\mathcal{G}^h = \mathcal{G}^h_v \times \mathrm{U}(1) \cong H^1(X;\mathbb{Z}) \times \mathrm{U}(1) = \mathbb{Z}^{b_1(X)} \times \mathrm{U}(1)$$

5.3 The Structure of the Base Space

We now turn our attention to $\mathcal{B} \cong \mathcal{C}_{\mathfrak{s}}/\mathcal{G}$. Using our discussion of $\mathcal{G}$, we will do so by studying $\mathcal{C}_{\mathfrak{s}}/\mathcal{G}^{\perp}$ and then moving on to consider $(\mathcal{C}_{\mathfrak{s}}/\mathcal{G}^{\perp})/\mathcal{G}^h$.

Theorem 5.26. *The action of $\mathcal{G}^{\perp}$ on $\mathcal{C}_{\mathfrak{s}}$ admits a global slice S, i.e. $\mathcal{C}_{\mathfrak{s}}$ is diffeomorphic to $S \times \mathcal{G}^{\perp}$.*

Proof. Fix a $U(1)$-connection A_0. Let S be the affine space

$$S = \left\{(A_0 + a, \Phi) \in \mathcal{C}_{\mathfrak{s}} \,\middle|\, a \in i\Omega^1(X) \text{ with } \mathrm{d}^* a = 0\right\}$$

We may consider $\mathrm{d}^* a$ as a "gauge fixing condition", since

$$\begin{aligned} \mathfrak{e} : \mathcal{G}^{\perp} \times S &\longrightarrow \mathcal{C}_{\mathfrak{s}} \\ (e^{if}, (A_0 + a, \Phi)) &\longmapsto (A_0 + a - i\mathrm{d}f, e^{if}\Phi) \end{aligned}$$

is a diffeomorphism with inverse

$$\begin{aligned} \mathfrak{e}^{-1} : \mathcal{C}_{\mathfrak{s}} &\longrightarrow \mathcal{G}^{\perp} \times S \\ (A_0 + b, \Psi) &\longmapsto (e^{-G(\mathrm{d}^* b)}, (A_0 + b - \mathrm{d}(G(\mathrm{d}^* b)), e^{G(\mathrm{d}^* b)}\Psi)) \end{aligned}$$

We note that $\mathfrak{e}^{-1}$ is defined in a sensible way: $e^{if} \in \mathcal{G}^{\perp}$ if $\int_X f \mathrm{vol}_g = 0$, which is equivalent to $H(f) = 0$ since $f \in (\mathcal{H}^0(X))^{\perp}$ if and only if $\langle f, g\rangle_{L^2} = 0$ for every constant $g \in \mathcal{H}^0(X)$, but constant functions make up $\mathcal{H}^0(X)$. Thus, to see that $\exp(-G(\mathrm{d}^* b)) \in \mathcal{G}^{\perp}$, we just need $HG(\mathrm{d}^* b)) = 0$, but $HG = GH$ and therefore it suffices that $\mathrm{d}^* b \in (\mathcal{H}^0(X))^{\perp}$, which we already established.

Similar reasoning shows that $\mathrm{d}^*(b - \mathrm{d}(G(\mathrm{d}^* b))) = \mathrm{d}^* b - (H + \Delta G(\mathrm{d}^* b)) = 0$. Now, one should check that $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ are indeed inverse to each other. this is left as an exercise to the reader. □

Hence, $\mathcal{C}_{\mathfrak{s}}/\mathcal{G}^{\perp} \cong S$, an affine space. We need to now consider $S/\mathcal{G}^h$: the group $\mathcal{G}^h$ does not act freely on S, but it does act freely on $S^* := S \cap \mathcal{C}^*_{\mathfrak{s}}$.

Theorem 5.27. *The quotient space $\mathcal{B}^* := \mathcal{C}^*_{\mathfrak{s}}/\mathcal{G} = S^*/\mathcal{G}^h$ is a smooth Hilbert manifold with the weak homotopy type of $\mathbb{CP}^{\infty} \times T^{b_1(X)}$, where $T^{b_1(X)}$ is the torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.*

Sketch of Proof. $\mathcal{G}^h \cong \mathcal{G}^h_v \times \mathrm{U}(1)$ acts freely on S^* and $\mathcal{G}^h_v$ acts properly discontinuously, hence $S^*/\mathcal{G}^h_v$ is Hausdorff. Compactness of the remaining $\mathrm{U}(1)$ then guarantees that $S^*/\mathcal{G}^h$ is Hausdorff as well.

Remark 5.28. Since $\mathcal{M}_{\omega} \subset \mathcal{B}^*$ is, near an irreducible configuration, a subspace of $\mathcal{B}^*$, it is also Hausdorff

Since there exist explicit local slices for $\mathcal{G}^h \curvearrowright S^*$, $\mathcal{B}$ is a Hilbert manifold. S is, as an affine space, contractible. But what about S^*? We claim without proof that $S \setminus S^*$ has infinite codimension in S and that this implies that S^* is weakly contractible. It follows that $\mathcal{B}^* = S^*/\mathcal{G}^h$ is a weak classifying space for $\mathcal{G}^h = U(1) \times H^1(X;\mathbb{Z})$, which has classifying space $\mathbb{CP}^\infty \times T^{b_1(X)}$. This means that $\mathcal{B}^*$ also has this weak homotopy time. $\square$

Corollary 5.29. *Let $\widetilde{\mathcal{B}^*} = S^*/\mathcal{G}^h_v$. Then the projection $\widetilde{\mathcal{B}^*} \to \mathcal{B}^*$ is a principal S^1-bundle.*

As another consequence of the theorem, note that since the Euler class $e \in H^2(\mathcal{B}^*;\mathbb{Z})$ restricted to $\mathbb{CP}^\infty$ is a generator of $H^2(\mathcal{B}^*;\mathbb{Z})$, the cohomology ring of $\mathcal{B}^*$ is

$$H^*(\mathcal{B}^*;\mathbb{Z}) = \mathbb{Z}[e] \otimes H^*(T^{b_1(X)};\mathbb{Z}) = \mathbb{Z}[e] \otimes \Lambda^*[a_1, \dots, a_{b_1(X)}]$$

5.4 Reducible Solutions and the Parameter Space

Recall that the reducible solutions are of the form $(A, 0) \in \mathcal{Z}_\omega$. The Dirac equation then becomes trivial: We just need to solve the curvature equation $F^+_{\hat{A}} = \omega$. Since we want to *avoid* reducible solutions, we are interested in the following question: Given some (parameter) $\omega \in i\Omega^2_+(X)$, does there exist a connection A with $F_{\hat{A}} = \omega$?

Lemma 5.30. *There exists a well-defined pairing*

$$\begin{aligned} H^2_{dR}(X) \times \mathcal{H}^2_+(X) &\longrightarrow \mathbb{R} \\ ([\mu], \tau) &\longmapsto \langle [\mu], \tau \rangle_{L^2} = \int_X \mu \wedge *\tau = \int_X \mu \wedge \tau \end{aligned}$$

Proof. Adding an exact form $\mathrm{d}\alpha$ to μ does not change its integral, since $\mathrm{d}\alpha \wedge \tau = \mathrm{d}(\alpha \wedge \tau)$. $\square$

Now let g be a Riemannian metric on X, and $L \to X$ a complex line bundle.

Definition 5.31. We set $\mathcal{W}_{g,L} = \{\omega \in i\Omega^2_+(X) \mid |\langle \omega + 2\pi c_1(L), \mathcal{H}^2_+(X)\rangle \equiv 0\}$.

Lemma 5.32. *$\mathcal{W}_{g,L}$ is an infinite-dimensional affine subspace of $i\Omega^2_+(X)$ of codimension $b_2^+(X) = \dim \mathcal{H}^2_+(X)$.*

Proof. Being an element of $\mathcal{W}_{g,L}$ imposes $b_2^+(X)$ independent conditions on ω. $\square$

Theorem 5.33. *For a given complex line bundle $L \to X$ and a Riemannian metric g there exists a* $\mathrm{U}(1)$*-connection A on L with $F^+_A = \omega$ if and only if $\omega \in \mathcal{W}_{g,L}$.*

The proof of this theorem relies on the following result.

Lemma 5.34. *Let $\beta \in \Omega^2_+(X)$. Then there exists some $\alpha \in \Omega^1(X)$ with $\beta = (\mathrm{d}\alpha)^+$ if and only if $\langle \beta, \mathcal{H}^2_+(X)\rangle \equiv 0$.*

Proof. Using Hodge decomposition as in proposition 3.55, one sees that $\Omega^2_+(X) = \mathcal{H}^2_+(X) \oplus (\mathrm{d}\Omega^1(X))^+$. $\square$

Proof of Theorem. Let A_0 be any connection on L. Then $A = A_0 + a$ where $a \in i\Omega^1(X)$ has curvature $F_A = F_{A_0} + \mathrm{d}a + a \wedge_{\text{comp}} a = F_{A_0} + \mathrm{d}a$ since $a \wedge a$ is zero for a form with coefficients in a commutative ring (e.g. in $\operatorname{End} L = i\mathbb{R}$). Therefore $F^+_A = F^+_{A_0} + \mathrm{d}^+ a$ and $F^+_A = \omega$ precisely if we can solve the equation $\mathrm{d}^+ a = \omega - F^+_{A_0} = (\omega - F^+_{A_0})^+$.

By the previous lemma, this is possible precisely if $\langle \omega - F_{A_0}, \mathcal{H}^2_+(X)\rangle_{L^2} = 0$ (we added back the anti-self dual part since it does not contribute in any case). From Chern-Weil theory, we know that $[F_{A_0}] = -2\pi i c_1(L)$, hence the ω-perturbed curvature equation has a solution if and only if $\langle \omega + 2\pi i c_1(L), \mathcal{H}^2_+(X)\rangle = 0$, which means that $\omega \in \mathcal{W}_{g,L}$. $\square$

The corollaries of the theorem above can be best understood in light of the following definition.

Definition 5.35. For a Spin^c-structure $\mathfrak{s}$ on X with characteristic line bundle L, we define the *wall* as the set $\mathcal{W}_{\mathfrak{s}} = \{(g, \omega) \in \mathcal{P} \mid \omega \in \mathcal{W}_{g,L}\}$. It is an infinite-dimensional submanifold of $\mathcal{P}$ of codimension $b_2^+(X)$. The connected components of $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ are called *chambers*.

Corollary 5.36. *The SW equations for parameters $(g, \omega) \in \mathcal{P}$ have a reducible solution if and only if $(g, \omega) \in \mathcal{W}_{\mathfrak{s}}$.*

Corollary 5.37. *Depending on the value of $b_2^+(X)$, we have the following cases.*

(i) *If $b_2^+(X) = 0$, then for all $(g, \omega) \in \mathcal{P}$, there are reducible solutions (more about this in the proof of Donaldson's theorem).*

(ii) *In case $b_2^+(X) = 1$, one generically finds oneself outside the wall, but parameters (g_1, ω_1) and (g_2, ω_2) can only be connected by a path avoiding the wall if they are in the same chamber.*

(iii) *For $b_2^+(X) \geq 2$, paths transverse to the wall are actually disjoint from the wall, hence parameters in the complement of the wall can always be connected by a curve avoiding the wall, i.e. there is only one chamber.*

We will see later that this corollary implies that the SW invariants are independent of parameters if $b_2^+(X) \geq 2$. For $b_2^+(X) = 1$, one has to deal with so-called "wall-crossing" phenomena.

5.5 Transversality and the Moduli Space

In this section, we build on the discussion of section 5.1.2 where, under the assumption $H^2_{(A,\Phi)} = 0$, we showed that $\mathcal{M}^*_\omega$ locally looks like a manifold of dimension $c_2(V_+)$. In this section, we establish:

Theorem 5.38. *For any fixed Riemannian metric g on X, and a generic $\omega \in i\Omega^2_+(X)$, the irreducible part of the moduli space $\mathcal{M}^*_\omega = (\mathcal{Z}_\omega \cap \mathcal{C}^*_{\mathfrak{s}})/\mathcal{G}$, is either a smooth manifold of dimension $c_2(V_+)$, or empty.*

If we vary the self-dual form that appears in the perturbed curvature equation, we get a collection of moduli spaces; it is useful to collect all this information into a single object, which leads the following definition:

Definition 5.39 (Parametrized Moduli Space)**.** We define the parametrized moduli space as

$$\mathcal{M} = \left\{([A, \Phi], \omega) \in \mathcal{B} \times i\Omega^2_+(X) \;\middle|\; f_\omega(A, \Phi) = 0\right\}$$

We also define its irreducible part as

$$\mathcal{M}^* = \mathcal{M} \cap \left(\mathcal{B}^* \times i\Omega^2_+(X)\right)$$

If $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ is the canonical projection, observe that $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$. The use of the parametrized moduli space is demonstrated by the following theorem:

Theorem 5.40 (Transversality)**.**

(i) *$\mathcal{M}^*$ is a Banach manifold.*

(ii) *The projection $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ is Fredholm, with index $c_2(V_+)$.*

(iii) *The set of regular values $\omega \in i\Omega^2_+(X)$ of π is generic. For a regular value ω, $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$ is either a smooth manifold of dimension $c_2(V_+)$, or empty.*

Proof.

(i) Pick a local slice S^* for the action $\mathcal{G} \curvearrowright \mathcal{C}^*_{\mathfrak{s}}$ around $(A_0, \Phi_0) \in \mathcal{Z}_\omega$. Consider the map

$$\begin{aligned} F: S^* \times i\Omega^2_+(X) &\longrightarrow i\Omega^2_+ \times \Gamma(V_-) \\ (A, \Phi, \omega) &\longmapsto (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi) \end{aligned}$$

It is clear from lemma 5.4 that its differential is given by

$$\mathcal{T}_{(A,\Phi,\omega)} F(a, \varphi, \tau) = (2\mathrm{d}^+ a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi) - \tau, D^+_A \varphi + \gamma(a)\Phi)$$

Our goal is to show that if (A, Φ) solve the ω-perturbed monopole equations (i.e. $F(A, \Phi, \omega) = 0$), $\mathcal{T}_{(A,\Phi,\omega)} F$ is surjective.

Let $(\eta, \psi) \in i\Omega^2_+(X) \times \Gamma(V_-)$ be orthogonal to the image of $\mathcal{T}_{(A,\Phi,\omega)} F$, i.e.

$$\int_X (\langle 2\mathrm{d}^+ a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi) - \tau, \eta\rangle + \langle D^+_A \varphi + \gamma(a)\Phi, \psi\rangle)\mathrm{vol}_g = 0$$

for every (a, φ, τ). Our task is to show that $(\eta, \psi) = (0, 0)$. Note that the second term does not depend on τ, though the first one does. Since we may choose τ at will, η must vanish. We are left with the equation

$$\int_X \langle D^+_A \varphi, \psi\rangle \mathrm{vol}_g = \int_X \langle \gamma(a)\Phi, \psi\rangle \mathrm{vol}_g$$

Now note that the right-hand side does not depend on φ, hence

$$\int_X \langle D^+_A \varphi, \psi\rangle \mathrm{vol}_g = \int_X \langle \varphi, D^-_A \psi\rangle \mathrm{vol}_g$$

does not depend on φ. Since we may choose φ arbitrarily, we conclude that $D^-_A \psi = 0$. Finally, we have

$$\int_X \langle \gamma(a)\Phi, \psi\rangle \mathrm{vol}_g = 0$$

for any $a \in i\Omega^1(X)$. By assumption, $\Phi \not\equiv 0$. Thus, consider a point p such that $\Phi(p) \neq 0$. Choose $a \in \Omega^1(X)$ locally such that $\gamma(a(p))\Phi(p) = \psi(p)$. If $\psi(p) \neq 0$, the integrand will be positive near p, and using a cutoff function we can force the integral to be strictly positive. This is a contradiction, hence $\psi(p) = 0$. Since Φ is nonzero on an open neighborhood U of p, we conclude $\psi|_U \equiv 0$. A general property of elliptic operators (such as D^-_A) now implies that $\psi \equiv 0$ since $D^-_A \psi = 0$. Thus, $\mathcal{T}_{(A,\Phi,\omega)} F$ is surjective and $(0, 0)$ is a regular value of F. As the preimage of a regular value, $\mathcal{M}^*$ is a Banach manifold.

(ii) We will show that (for $[(A, \Phi)] \in \mathcal{M}^*$—we will omit the square brackets in the rest of this proof) the kernel and cokernel of $\mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*}$ (meaning we consider only tangent vectors to $\mathcal{M}^*$) coincide with the kernel and cokernel of $\mathcal{T}_{(A,\Phi)} f_\omega|_S = \mathcal{T}_{(A,\Phi)} f_\omega|_{S^*}$ (since $(A, \Phi) \in \mathcal{C}^*_{\mathfrak{s}}$, $S = S^*$ near (A, Φ)), which in turn are simply $H^1_{(A,\Phi)}$ and $H^2_{(A,\Phi)}$ (cf. lemma 5.9).

Clearly $\mathcal{T}_{(A,\Phi,\omega)} F = \mathcal{T}_{(A,\Phi)} f_\omega|_S - (\tau, 0)$, hence

$$T_{(A,\Phi,\omega)} \mathcal{M}^* = \ker \mathcal{T}_{(A,\Phi,\omega)} F = \{(a, \varphi) \mid \mathcal{T}_{(A,\Phi)} f_\omega(a, \varphi) \in TS^* = (\tau, 0)\}$$

and $\mathcal{T}_{(A,\Phi,\omega)} \pi(a, \varphi, \tau) = \tau$, therefore $(a, \varphi, \tau) \in \ker \mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*}$ precisely if $\tau = 0$ and $(a, \varphi) \in \ker T_{(A,\Phi)} f_\omega|_S$ (the latter condition simply expresses that $(a, \varphi, \tau) \in T_{(A,\Phi,\omega)} \mathcal{M}^*$). We conclude that $\ker \mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*} \cong \ker \mathcal{T}_{(A,\Phi)} f_\omega|_{S^*} \cong H^1_{(A,\Phi)}$.

Given $(\eta, \psi) \in i\Omega^2_+(X) \times \Gamma(V_-)$, we can use the fact that $(0, 0)$ is a regular value of F to see that we can always find (a, φ, τ) such that $\mathcal{T}_{(A,\Phi)} f_\omega(a, \varphi) = (\eta + \tau, \varphi)$. Therefore, any element of $\operatorname{coker} \mathcal{T}_{(A,\Phi)} f_\omega$ is of the form $(\mu, 0)$. This means that the composite map

$$\begin{aligned} \kappa: i\Omega^2_+(X) &\longrightarrow i\Omega^2_+ \times \Gamma(V_-) \xrightarrow{\text{proj}} H^2_{(A,\Phi)} \\ \mu &\longmapsto (\mu, 0) \longmapsto [(\mu, 0)] \end{aligned}$$

is surjective. Thus, $H^2_{(A,\Phi)} \cong i\Omega^2_+ / \ker\kappa$. The kernel of κ consists of elements represented by $(\mu, 0) \in \operatorname{im} \mathcal{T}_{(A,\Phi)} f_\omega|^*_S$, which is equivalent to the existence of some tuple $(a, \varphi, \mu) \in \mathcal{T}_{(A,\Phi,\omega)}\mathcal{M}^*$. But this precisely means $\mu \in \operatorname{im} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mu M^*}$. Thus, $\ker\kappa \cong \operatorname{im} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mu M^*}$ and $H^2_{(A,\Phi)} \cong \operatorname{coker} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mathcal{M}^*}$.

(iii) This is a result in the spirit of Sard's theorem; we will not prove it.

□

5.5.1 Compactness of the Moduli Space

When talking about compactness of the moduli space, we will always mean *sequential* compactness, i.e. any sequence has a convergent subsequence. The proof of this requires the following deep result, which we do not prove:

Theorem 5.41 (Sobolev Embedding)**.** *Let X be a compact n-manifold. Then*

- *There exists an embedding $L^p_{j+m}(X) \subset C^j(X)$ if $mp \geq n$.*
- *This embedding is compact if $mp > n$.*

Corollary 5.42. *Let X be a compact 4-manifold. Then there exists a compact embedding $L^2_k(X) \subset C^{k-3}(X)$. Hence, every bounded sequence in $L^2_k(X)$ has a convergent subsequence in C^{k-3}.*

Proof. Here, $n = 4$, $p = 2$ and $m = 3$. □

The precise statement that we will prove is:

Theorem 5.43. *Let (A_i, Φ_i) be a sequence of L^2_4 solutions to the SW equations. Then there exists a sequence u_i of L^2_5 gauge transformations such that $u_i(A_i, \Phi_i)$ is a bounded sequence in L^2_k for all k. Hence the solutions $u_i(A_i, \Phi_i)$ are C^∞ and there is a subsequence that converges in the C^∞ topology to a C^∞ solution (A, Φ) of the monopole equations. In particular, the moduli space is sequentially compact in the C^∞ topology.*

We start by proving some pointwise bounds:

Lemma 5.44. *The Laplacian of $|\Phi|^2$ can be expressed as*

$$\frac{1}{2}\Delta|\Phi|^2 = \mathrm{Re}\langle \nabla_A^* \nabla_A \Phi, \Phi\rangle - |\nabla^A \Phi|^2$$

Proof. In a local orthonormal frame for TX, and $f \in C^\infty(M)$, the Laplacian is given by $\nabla f = -\sum_i \nabla^2_{e_i,e_i} f$, where $\nabla^2_{X,Y} = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f = L_X L_Y f - L_{\nabla_X Y} f$. Using $f = |\Phi|^2$, we find:

$$\begin{aligned}
\frac{1}{2}\nabla|\Phi|^2 = -\frac{1}{2}\nabla^2_{e_i,e_i}|\Phi|^2 &= -\sum_i \Big(\nabla_{e_i} \mathrm{Re}(\langle \nabla^A_{e_i}\Phi, \Phi\rangle) - \mathrm{Re}(\langle \nabla^A_{\nabla_{e_i} e_i}\Phi, \Phi\rangle)\Big) \\
&= -\sum_i \mathrm{Re}(\langle \nabla^A_{e_i}\nabla^A_{e_i}\Phi, \Phi\rangle + \langle \nabla^A_{e_i}\Phi, \nabla^A_{e_i}\Phi\rangle - \langle \nabla^A_{\nabla_{e_i} e_i}\Phi, \Phi\rangle)\Big)
\end{aligned}$$

All that is left to show is that the first and last term combine as follows:

$$-\sum_i \Big(\mathrm{Re}(\langle \nabla^A_{e_i}\nabla^A_{e_i}\Phi, \Phi\rangle - \langle \nabla^A_{\nabla_{e_i} e_i}\Phi, \Phi\rangle)\Big) = \mathrm{Re}\langle \nabla^*_A \nabla_A \Phi, \Phi\rangle$$

But this just the real part of the statement of lemma 4.10. □

This yields a useful bound on $|\Phi|^2$:

Lemma 5.45. *Consider an L^2_4-solution (A, Φ) to the monopole equations and let $p \in X$ be a maximum of $|\Phi|^2$. Then $|\Phi(p)|^4 \leq -s_g(p)|\Phi(p)|^2$.*

Proof. At a maximum, $\Delta|\Phi|^2 \geq 0$. Using our lemma, we find (in p):

$$0 \leq \mathrm{Re}\langle \nabla_A^* \nabla_A \Phi, \Phi\rangle - |\nabla^A \Phi|^2 \leq \mathrm{Re}\langle \nabla_A^* \nabla_A \Phi, \Phi\rangle$$

The Weitzenböck formula (theorem 4.8) tells us

$$0 \leq \mathrm{Re}(\langle D_A^- D_A^+ \Phi, \Phi\rangle - \frac{1}{2}\langle \gamma(F_{\hat{A}}^+)\Phi, \Phi\rangle) - \frac{1}{4} s_g |\phi|^2$$

Using that (A, Φ) is a solution to the SW equations as well as lemma 4.23 turns this into:

$$0 \leq -\frac{1}{4} s_g |\Phi|^2 - \frac{1}{2}\langle(\gamma(\sigma(\Phi,\Phi))\Phi, \Phi\rangle = -\frac{1}{4} s_g |\Phi|^2 - 2|\sigma(\Phi,\Phi)|^2 = -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4)$$

whence we obtain $|\Phi(p)|^4 \leq -s_g(p)|\Phi(p)|^2$. □

Corollary 5.46. *If $s_g \geq 0$ everywhere on X, then $\Phi \equiv 0$, i.e. every solution to the monopole equations is reducible. If $s_g < 0$ somewhere on X, then*

$$|\Phi|^2 \leq \max_{p \in X} -s_g(p) > 0$$

Proof. In the first case, the left hand side is non-negative while the right hand side is non-positive: Both must therefore vanish. The second bound holds in the point p, which is a maximum of $|\Phi|^2$. Therefore it holds everywhere. □

This is a C^0-bound on $|\Phi|^2$. Notice that since $|F_{\hat{A}}^+| = |\sigma(\Phi,\Phi)| = |\Phi|^2/2\sqrt{2}$ (by the curvature equation and lemma 4.23), we automatically obtain a C^0-bound for $|F_{\hat{A}}^+|$.

Now, we look for analogous results for the perturbed SW equations. Let $p \in X$ be a maximum of $|\Phi|^2$ (with (A, Φ) a solution); going through the same steps, we obtain an extra term:

$$0 \leq \frac{1}{2}\Delta|\Phi|^2 \leq -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) - \frac{1}{2}\langle\gamma(\omega)\Phi, \Phi\rangle$$

Using lemma 4.23 and the Cauchy-Schwarz inequality, we find

$$0 \leq -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) - 2\langle\omega, \sigma(\Phi,\Phi)\rangle \leq -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) + 2|\omega||\sigma(\Phi,\Phi)| = -\frac{|\Phi|^2}{4}(s_g + |\Phi|^2 - 2\sqrt{2}|\omega|)$$

We find that in p, $|\Phi|^4 \leq -|\Phi|^2(s_g - 2\sqrt{2}|\omega|)$. This shows:

Proposition 5.47. *Let $s_{g,\omega} = \min_{p \in X}\{0, s_g(p) - 2\sqrt{2}|\omega(p)|\}$ (notice that it depends only on $(g,\omega) \in \mathcal{P}$), and let $(A, \Phi) \in \mathcal{Z}_\omega$. Then we have the pointwise bounds*

$$|\Phi|^2 \leq -s_{g,\omega}$$
$$|F_{\hat{A}}^+| \leq -\frac{1}{2\sqrt{2}} s_{g,\omega} + \max_{p \in X}|\omega|$$

where we used that $\omega \in L_3^2$, which embeds in C^0, to see that it assumes its maximum.

We now turn to integral bounds.

Proposition 5.48. *Let (A, Φ) be a solution to the monopole equations. Then*

$$\|\Phi\|_{L^4}^2 \leq \left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2}$$
$$\left\|\nabla^A \Phi\right\|_{L^2} \leq \frac{1}{2}\left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2}$$
$$\left\|F_{\hat{A}}^+\right\|_{L^2} \leq \frac{1}{2\sqrt{2}}\left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2} + \|\omega\|_{L^2}$$

Proof. The first bound follows from the Weitzenböck formula and the usual spinor identities:

$$0 = \int_X \langle D_A^- D_A^+ \Phi, \Phi\rangle \mathrm{vol}_g = \int_X (|\nabla^A \Phi|^2 + \frac{1}{4}s_g|\Phi|^2 + \frac{1}{2}\langle \gamma(F_{\hat{A}}^+)\Phi, \Phi\rangle)\mathrm{vol}_g$$
$$= \int_X (|\nabla^A \Phi|^2 + \frac{1}{4}s_g|\Phi|^2 + \frac{1}{4}|\Phi|^4 + 2\langle \omega, \sigma(\Phi,\Phi)\rangle)\mathrm{vol}_g$$

Discarding the first term and using the Cauchy-Schwarz inequality twice (in different incarnations), we have:

$$\int_X |\Phi|^4 \mathrm{vol}_g \leq -\int_X (s_g|\Phi|^2 + 8\langle \omega, \sigma(\Phi,\Phi)\rangle)\mathrm{vol}_G$$
$$\leq \int_X (-s_g|\Phi|^2 + 8|\omega||\sigma(\Phi,\Phi)|)\mathrm{vol}_g = \int_X |\Phi^2|(-s_g + 2\sqrt{2}|\omega|)\mathrm{vol}_g$$
$$\leq \left(\int_X |\Phi|^4 \mathrm{vol}_g\right)^{1/2} \left(\int_X (-s_g + 2\sqrt{2}|\omega|)^2 \mathrm{vol}_g\right)^{1/2}$$

Thus, we obtain

$$\|\Phi\|_{L^4}^2 \leq \|-s_g + 2\sqrt{2}|\omega|\|_{L^2}$$

as promised. For the second inequality, discard the $|\Phi|^4$-term instead of the $\nabla^A\Phi$-term and proceed analogously. For the last bound, we once again need Cauchy-Schwarz:

$$\int_X |F_{\hat{A}}^+|^2 \mathrm{vol}_g = \int_X |\sigma(\Phi,\Phi) + \omega|^2 \mathrm{vol}_g = \int_X \left(\frac{1}{8}|\Phi|^4 + 2\langle \sigma(\Phi,\Phi), \omega\rangle + |\omega|^2\right)\mathrm{vol}_g$$
$$\leq \int_X \left(\frac{1}{8}|\Phi|^2 + \frac{1}{\sqrt{2}}|\Phi|^2|\omega| + |\omega|^2\right)\mathrm{vol}_g = \int_X \left(\frac{1}{2\sqrt{2}}|\Phi|^2 + |\omega|\right)^2 \mathrm{vol}_g = \left\|\frac{1}{\sqrt{8}}|\Phi|^2 + |\omega|\right\|_{L^2}^2$$

Taking the square root and using the triangle inequality as well as our first bound, we obtain the required result:

$$\|F_{\hat{A}}^+\|_{L^2} \leq \frac{1}{2\sqrt{2}}\|-s_g + 2\sqrt{2}|\omega|\|_{L^2} + \|\omega\|_{L^2}$$

□

We are still far from completing our theorem but we can already draw some important conclusions:

Corollary 5.49. *Let X be a closed, oriented 4-manifold. For fixed parameters $(g,\omega) \in \mathcal{P}$, there exist at most finitely many* Spin^c*-structures on X such that $\mathcal{M}_\omega \neq \varnothing$ and $\dim^{\exp}\mathcal{M}_\omega \geq 0$.*

Proof. Recall that $\dim^{\exp}\mathcal{M}_\omega \geq 0$ is equivalent to $c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X)$. We saw in the proof of proposition 4.22 that Chern-Weil theory implies $4\pi^2 c_1^2(L_\mathfrak{s}) = \|F_{\hat{A}}^+\|_{L^2}^2 - \|F_{\hat{A}}^-\|_{L^2}^2$ and therefore our L^2-bound on $F_{\hat{A}}^+$ implies an L^2-bound from above on $F_{\hat{A}}^-$, depending only on (g,ω).

Now, we have bounded $c_1^2(L_\mathfrak{s})_\mathbb{R} \in H^2(X;\mathbb{R})$ using a bound that depends only on (g,ω). Since $c_1(L_\mathfrak{s}) \in H^2(X;\mathbb{Z})$ and the free part of $H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{R})$ is a lattice while the torsion is finite, only finitely different first Chern classes are possible. □

Corollary 5.50. *Assume $b_2^+(X) > 0$, and $(g,\omega) \in \mathcal{P}$ is generic. Then there are at most finitely many* Spin^c*-structures with $\mathcal{M}_\omega \neq \varnothing$.*

Proof. Under the given assumptions, there are no reducible solutions, and solutions are transverse zeros of f_ω. Hence $\mathcal{M}_\omega$ is a smooth manifold and $\dim^{\exp}\mathcal{M}_\omega \geq 0$. □

To establish compactness we need bounds on $\|A\|_{L_k^2}$, and $\|\Phi\|_{L_k^2}$ (up to gauge transformation) that depend only on (X,g) and k. To begin with note that all of the bounds obtained above depend only on (X,g) in the case of the unperturbed SW equations; we now restrict to this case.

The three following results are "black boxes", i.e. outside the scope of this lecture, but they are nevertheless needed to prove theorem 5.43. In the following, we will use c, c' to denote several (different) constants whose values are of no importance.

Theorem 5.51 (Elliptic Estimate)**.** *The Dirac operator is elliptic and satisfies*

$$\|\Phi\|_{L^p_{k+1}} \leq c\left(\|D^+_{A_0}\Phi\|_{L^p_k} + \|\Phi\|_{L^p}\right)$$

Theorem 5.52. *Let P denote the L^2-projection onto the kernel of a linear elliptic first order differential operator $\mathfrak{L}$ (for our purposes, $\mathfrak{L} = D_{A_0}$ and $\mathfrak{L} = \mathrm{d} \oplus \mathrm{d}^*$ are relevant). Then*

$$\|\Phi - P\Phi\|_{L^p_{k+1}} \leq c\|\mathfrak{L}\Phi\|_{L^p_k}$$

Theorem 5.53 (Gauge Fixing)**.** *Let $E \to X$ be a complex line bundle and $\hat{A}_0$ a fixed, smooth $\mathrm{U}(1)$-connection. Up to L^2_{k+2} gauge transformations we can write an arbitrary L^2_{k+1} connection $\hat{A}$ as $\hat{A} = \hat{A}_0 + a$, with $\mathrm{d}^*a = 0$ and*

$$\|a\|^2_{L^2_{k+1}} \leq c\left\|F^+_{\hat{A}}\right\|^2_{L^2_k} + c'$$

The first monopole equation can be rewritten as

$$D^+_{A_0}\Phi = -\gamma(a)\Phi$$

and the above theorems then allow us to deduce the following important result:

Corollary 5.54 (Bootstrapping)**.** *Suppose a, Φ are bounded in L^2_3 by constants c, c'. Then they are bounded by $c(k)$, $c'(k)$ in L^2_k for all $k \geq 3$.*

Proof. A Sobolev multiplication theorem asserts that the multiplication $L^2_k \times L^2_k \to L^2_k$ for every $k \geq 3$ is bounded. Assuming L^2_3-bounds on a, Φ, we obtain bounds on $\gamma(a)\Phi$ and $\sigma(\Phi,\Phi)$ and therefore on $D^+_{A_0}\Phi$ and $F^+_{\hat{A}}$. Using the elliptic estimate, Φ is then also bounded in L^2_4. By gauge fixing, a is bounded in L^2_4 as well. Now we can go through the same steps and inductively obtain bounds for any $k \geq 3$. □

All that is left to establish compactness of the moduli space is:

Theorem 5.55. *There exist constants c, c' depending only on (X, g) such that any solutions of the SW equations is gauge equivalent to a solution (A, Φ) with $A = A_0 + a$, $\mathrm{d}^*a = 0$ and*

$$\|a\|_{L^2_3} \leq c$$
$$\|\Phi\|_{L^2_3} \leq c'$$

To prove it, we start with the observation that our L^2-bound on $F^+_{\hat{A}}$ yields an L^2_1-bound on a. To get an L^2_2-bound, we use:

Proposition 5.56. *There exists a constant c depending only on (X, g) such that for any $(A, \Phi) \in \mathcal{Z}_\omega$, $\left\|F_{\hat{A}}\right\|^2_{L^2_1} \leq c$.*

Proof. Using $F^+_{\hat{A}} = \sigma(\Phi,\Phi)$, combined with the fact that we have both an L^∞-bound on Φ and an L^2-bound on $\nabla^A\Phi$, we get an L^2-bound on $\nabla F^+_{\hat{A}}$, where ∇ is induced by the Levi-Cività connection. Thus, we have an L^2-bound on $\mathrm{d}F^+_{\hat{A}}$. □

We have now established:

Corollary 5.57. *There exist a constant c depending only on (X, g) such that any $(A, \Phi) \in \mathcal{Z}_\omega$ is gauge equivalent to a solution (A, Φ) with $A = A_0 + a$, $\mathrm{d}^*a = 0$, and $\|a\|^2_{L^2_2} \leq c$.*

Proof of Theorem. There is an L_1^2-bound on Φ since we have L^2-bounds on Φ, a and $\nabla^A\Phi$. Now we use Sobolev multiplication to see that $L_2^2 \times L^\infty \to L^4$, $(a,\Phi) \mapsto -\gamma(a)\Phi = D_{A_0}^+\Phi$ is bounded. The orthogonal projection theorem for $D_{A_0}^+$ then yields an L_1^4-bound on Φ.

Now, we use the same steps to get different L_k^p bounds. Every time, we need a Sobolev multiplication theorem. First, we have:

$$\begin{aligned} L_2^2 \times L_1^4 &\longrightarrow L_1^3 \\ (a,\Phi) &\longmapsto -\gamma(a)\Phi \end{aligned}$$

which yields an L_2^3-bound on Φ. Then

$$\begin{aligned} L_2^2 \times L_2^3 &\longrightarrow L_2^2 \\ (a,\Phi) &\longmapsto -\gamma(a)\Phi \end{aligned}$$

gives an L_3^2-bound. Finally, we find an L_3^2-bound on $F_{\hat{A}}^+$ by using the map

$$\begin{aligned} L_3^2 \times L_3^2 &\longrightarrow L_3^2 \\ (\Phi,\Phi) &\longmapsto \sigma(\Phi,\Phi) = F_{\hat{A}}^+ \end{aligned}$$

This establishes the L_3^2-bound on a (which is an element of $L_4^2(i\Omega^1(X))$). $\square$

6 Seiberg-Witten Theory

6.1 Donaldson's Theorem

6.1.1 Proof using Seiberg-Witten Theory

Donaldson's theorem was a landmark discovery in the theory of 4-manifolds. It highlights the differences between the topological and smooth categories.

Theorem 6.1 (Donaldson)**.** *If X is a closed, connected, oriented, smooth manifold with definite intersection form Q_X, then Q_X is diagonalizable over $\mathbb{Z}$.*

Proof. Without loss of generality, we may assume $b_2^+(X) = 0$ since we may swap orientation to make Q_X negative-definite. Our method of proof will be to add simplifying assumptions along the way until we derive a contradiction and subsequently backtrack to remove (or justify!) the assumptions one by one.

Assumption 1: $b_1(X) = 0$.

Fix a Riemannian metric g and a Spin^c-structure $\mathfrak{s}$; the latter exists by theorem 3.33. We consider now the SW equations for $(g, \mathfrak{s}, \omega) \in \mathcal{P}$. Since $b_2^+(X) = 0$, there always exists a reducible solution $(A, 0) \in \mathcal{Z}_\omega$. Finding this solution amounts to solving the curvature equation $F_{\hat{A}}^+ = \omega$. From Chern-Weil theory, we know that $[\frac{i}{2\pi} F_{\hat{A}}] = c_1(L_\mathfrak{s})$. Every closed 2-form representing this cohomology class is the curvature of a suitable A, since if A_0 is some fixed Spin^c-connection, the (self-dual) curvature of $\widehat{A_0 + a}$ is $F_{\hat{A}}^+ = F_{\hat{A}_0}^+ + 2\mathrm{d}^+ a$ so we must solve

$$2\mathrm{d}^+ a = \omega - F_{A_0}$$

Since $b_2^+(X) = 0$, $\mathrm{d}^+ : i\Omega^1 \to i\Omega^2_+$ is surjective (cf. section 3.4). Thus, we obtain a reducible solution and $\mathcal{M}_\omega \neq \varnothing$. In fact, it is unique up to gauge equivalence since if a, a' are solutions to the curvature equation, $a - a' \in i\Omega^1(X)$ is d^+-closed, hence d-closed, therefore exact because $b_1(X) = 0$ by assumption. However, exact forms can be gauged away. Recall that

$$\dim^{\exp} \mathcal{M}_\omega = \frac{1}{4}\left(c_1^2(L_\mathfrak{s}) - 2(\chi(X) + 3\sigma(X))\right)$$

It is a fact that $c_1(L_\mathfrak{s})$ is a lift of $w_2(X)$ if and only if $c_1^2(L_\mathfrak{s}) = \sigma(X) + 8k$ for some $k \in \mathbb{Z}$. (this is known as the *Van der Blij lemma*). Using $\sigma(X) = -b_2^-(X) = -b_2(X)$, we find:

$$\dim \mathcal{M}_\omega = \frac{1}{4}(\sigma(X) + 8k - 2(2 - 2b_1(X) + b_2^+(X) + b_2^-(X)) - 3\sigma(X)) = \frac{1}{4}(8k - 4) = 2k - 1$$

which is odd. Now, we add the next assumption:

Assumption 2: $k > 0$, equivalently, $\dim \mathcal{M}_\omega > 0$.

By transversality (see theorem 5.38), we may assume that $\mathcal{M}_\omega^*$ is smooth and of the expected dimension, since the reducible point (which sits inside a space of *positive* dimension) must be a deformation of irreducible solutions. Moreover $\mathcal{M}_\omega$ is compact with one singular point, corresponding to the reducible solution $(A, 0)$. Donaldson's original approach suggests that it is important to understand a neighborhood of the singular point. At $(A, 0)$, the linearized equations (cf. 5.6) decouple (crucially using $\Phi \equiv 0$) into

$$i\Omega^0 \cong \mathbb{R} \xrightarrow{-\mathrm{d}} i\Omega^1(X) = 0 \xrightarrow{\mathrm{d}^+} i\Omega^2_+ = 0$$

$$0 \xrightarrow{\qquad} \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-)$$

Recall from the generalized Atiyah-Singer index theorem that

$$\mathrm{ind}_\mathbb{C}\, D_A^+ = \frac{1}{8}(c_1^2(L_\mathfrak{s}) - \sigma(X)) = k > 0$$

We make our final assumption.

Assumption 3: D_A^+ is surjective.

In this case, $\operatorname{ind}_{\mathbb{C}} D_A^+ = \dim_{\mathbb{C}} \ker D_A^+$, i.e. $\ker D_A^+ \cong \mathbb{C}^k$. At the reducible point, the stabilizer subgroup of $\mathcal{G}$ is $\mathcal{G}_{(A,0)} = \mathrm{U}(1)$, given by the constant gauge transformations that rotate Φ (which now vanishes). Following the discussion after lemma 5.9, a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is given by $\psi^{-1}(0)/S^1$, where $\psi : H^1_{(A,0)} = \ker D_A^+ \to H^2_{(A,0)}$ is a choice of Kuranishi map. Our description of the stabilizer subgroup shows that the differential of ψ may fail to be surjective at $0 \in H^1_{(A,0)}$, but on $H^1_{(A,0)} \setminus \{0\}$, ψ is transverse to $0 \in H^2_{(A,0)}$. When assumption 3 is satisfied, a neighborhood of $[A, 0]$ in $\mathcal{M}_\omega$ is therefore a cone on $\mathbb{C}\mathrm{P}^{k-1}$: The cone point corresponds to the reducible solution.

We can cut out a neighborhood of the cone point, i.e. "truncate" $\mathcal{M}_\omega$ by removing an open cone around the singular point, we get a compact manifold with boundary $\mathbb{C}\mathrm{P}^{k-1}$. If k is *odd*, and this manifold is *orientable*, we obtain a contradiction since the truncated moduli space would demonstrate that $\mathbb{C}\mathrm{P}^{k-1}$ is null-cobordant, which is false since $\sigma(\mathbb{C}\mathrm{P}^{k-1}) \neq 0$ and the signature is a cobordism invariant.

Now, we begin the process of reconsidering our assumptions. If $\mathcal{M}^*_\omega$ is orientable, we also get a contradiction for k even, as we show now. Instead of looking at solutions modulo $\mathcal{G}$, we can look at solutions modulo $\mathcal{G}^\perp = \{u \in \mathcal{G} \mid u = e^{if} \ \int_X f \mathrm{vol}_g = 0\}$. Recall that $\mathcal{G} \cong \mathcal{G}^\perp \times \mathcal{G}^h$ and that $b_1(X) = 0$, hence $\mathcal{G} = \mathcal{G}_0$, i.e. every $u : X \to S^1$ is of the form $u = e^{if}$ for $f : X \to \mathbb{R}$. $u \in \mathcal{G}^h$ means that $u\mathrm{d}u^{-1} = -i\mathrm{d}f$ is harmonic. But then $\langle \mathrm{d}\mathrm{d}^*\mathrm{d}f, \mathrm{d}f\rangle = 0$, which implies that $\mathrm{d}^*\mathrm{d}f = 0$. Repeating this argument shows that $\mathrm{d}f = 0$, i.e. f is constant. This shows that $\mathcal{G}^h \cong U(1)$ in our situation.

It is now clear that away from the singular point $[A, 0]$, we have a circle bundle $\beta : \overline{\mathcal{M}}^*_\omega \to \mathcal{M}^*_\omega$. This space is a closed, oriented manifold and therefore has even Euler characteristic. On the other hand, as a space admitting a $U(1)$-action with a single fixed point, it has Euler characteristic ± 1, a contradiction. Regarding orientability of $\mathcal{M}^*_\omega$, it is a fact (which we will not prove) that $\mathcal{M}^*_\omega$ *is* actually orientable, so this "assumption" need not be relaxed.

Consider assumption 3: If it is not satisfied, we have a Kuranishi map

$$\psi : \mathbb{C}^{k+r} = H^1_{(A,0)} \to H^2_{(A,0)} = \mathbb{C}^r$$

It is still S^1-equivariant, with $\psi(0) = 0$ and $\psi \pitchfork 0 \in \mathbb{C}^r$ on $\mathbb{C}^{k+r} \setminus \{0\}$. A neighborhood of $[A, 0]$ in $\mathcal{M}_\omega$ is given by $\psi^{-1}(0)/S^1$. Moreover, ψ descends to a section of the vector bundle given by $H^2_{(A,0)}$ over $H^1_{(A,0)}/S^1 = C(\mathbb{C}\mathrm{P}^{k+r-1})$ ($C(X)$ denotes the cone over X). If $r = 1$, this would be the hyperplane bundle H but the constant gauge transformations making up S^1 act like p copies of the action on $\mathbb{C}$, i.e. we have a rank r bundle

$$H \oplus \cdots \oplus H \to C(\mathbb{C}\mathrm{P}^{k+r-1})$$

In the homology of $\mathbb{C}\mathrm{P}^{k+r-1}$, $\psi^{-1}(0)$ is Poincaré dual to x^r, where x is a generator of $H^2(\mathbb{C}\mathrm{P}^{k+r-1})$, i.e. for generic value of the cone parameter, $\psi^{-1}(0)$ is a smooth submanifold N with homology class of a linear $\mathbb{C}\mathrm{P}^{k-1} \subset \mathbb{C}\mathrm{P}^{k+r-1}$. This implies

$$\langle e(\beta)^{k-1}, [N]\rangle = \langle e(\beta)^{k-1}, [\mathbb{C}\mathrm{P}^{k-1}]\rangle \neq 0$$

and we obtain a contradiction with Stokes' theorem as before.

The first assumption can be maintained without loss of generality because of the following lemma:

Lemma 6.2. *If Q_X is the intersection form of a smooth manifold X, then it is (isomorphic over $\mathbb{Z}$ to) the intersection form of a smooth manifold Y with $b_1(Y) = k$ for any $k \in \mathbb{N}_0$.*

Proof. Raising the first Betti number without changing the intersection form is easy: $b_1(X\#k(S^1 \times S^3)) = b_1(X) + k$ and $S^1 \times S^3$ has no second cohomology, hence $Q_{X\#k(S^1\times S^1)} \cong Q_X$.

Lowering the first Betti number is done by means of surgery: Suppose $b_1(X) > 0$ and $H_1(X) \cong \mathbb{Z}^{b_1(X)} \oplus \text{Tor}$. Pick a basis element $\alpha \in H^1(X)$ for the free part: It can be realized by a smoothly embedded $S^1 \hookrightarrow X$. Pick a tubular neighborhood $T \cong S^1 \times D^3$ of the S^1 in X. Then

$$\partial T = \partial(S^1 \times D^3) = S^1 \times S^2 = \partial(D^2 \times S^2)$$

so we perform surgery: Remove T from X and glue in $B^2 \times S^2$ in its place. In the manifold Y obtained in this way, the circle we started with is null-homotopic, and therefore null-homologous. This shows that $b_1(Y) = b_1(X) - 1$. A Mayer-Vietoris argument shows that $H_2(X) \cong H_2(Y)$ and $Q(Y) \cong_{\mathbb{Z}} Q(X)$. □

Finally, we discuss the assumption $k > 0$:

Proposition 6.3 (Elkies)**.** *Let Q be a negative definite unimodular symmetric bilinear form over $\mathbb{Z}$. There exists a v with $v \cdot x \equiv x^2 \mod 2$ for all x and $v^2 > -\operatorname{rank} Q$ if and only if Q is not diagonalizable.*

Recall from proposition 3.30 that $w_2(X) \cdot \alpha = r(\alpha \cdot \alpha)$ (r denotes reduction modulo 2) for every $\alpha \in H^2(X; \mathbb{Z})$. Thus, if (and only if) Q_X is not diagonal over $\mathbb{Z}$, there exists some $c \in H^2(X; \mathbb{Z})$ with $r(c) = w_2(X) \in H^2(X; \mathbb{Z}_2)$ and $c^2 + b_2(X) = 8k > 0$. c determines a Spin^c structure $\mathfrak{s}$ with characteristic line bundle defined through by $c_1(L_\mathfrak{s}) = c$.

Fix $(g, \omega) \in \mathcal{P}$, and consider the SW equations for $(\mathfrak{s}, g\omega)$. As before $\mathcal{M}_\omega \neq \varnothing$ because there exists a reducible solution. The dimension is then the expected one and using $b_1(X) = 0$ and $c^2 = \sigma(X) + 8k = -b_2(X) + 8k$ we find:

$$\begin{aligned}\dim \mathcal{M}_\omega &= \frac{1}{4}(c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X)) = \frac{1}{4}(c^2 - (4 + 2b_2(X) - 3b_2(X))) \\ &= \frac{1}{4}(c^2 + b_2(X) - 4) = 2k - 1 > 0\end{aligned}$$

So outside the single singular point, $\mathcal{M}_\omega$ is a smooth manifold of positive dimension. Our earlier equation

$$0 \neq \langle w_2(\beta)^{k-1}, [N] \rangle$$

still holds, but β is zero-bordant as a bundle over the truncated moduli space. This final contradiction tells us that Q_X must be diagonal. □

6.1.2 Sketch of the Original Argument

The original proof, due to Donaldson, precedes Seiberg-Witten theory and is much more natural. It uses *instantons*. Let X be CCOS with $b_1(X) = 0$ and $b_2^+(X) = 0$ without loss of generality. Denote the space of "1-instantons" on X by $\mathcal{M}_1$. Let P be an $SU(2)$-principal bundle over X with $c_2(P) = 1$ and consider the anti self-duality equation $*F_A = -F_A$; its moduli space of solutions is $\mathcal{M}_1$. It turns out $\dim \mathcal{M}_1 = 5$ and that it *fails* to be compact. Understanding the non-compactness is key.

It turns out that the moduli space only has one "end", which looks like X itself since the non-compactness arises from concentrating arbitrarily much curvature at a single point: Such configurations are parametrized by X. Cut off the end an replace it with $X \times [0, \epsilon)$.

The bundle P splits as $P = L \oplus L^{-1}$ (i.e. $c_1(P) = 0$) and $c_2(P) = 1$ implies that $c_1^2(L) = -1$. Each such line bundle over X gives rise to a cone over $\mathbb{C}\mathrm{P}^2$ with a singular cone point, i.e. the number of singular points in $\mathcal{M}_1$ is equal to the number of classes (up to sign) $\pm c \in H^2(X; \mathbb{Z})$ such that $c_1^2 = -1$. Cutting away neighborhoods of the singular points, the truncated moduli space becomes an oriented cobordism between X and a disjoint union of $p$copies of $\mathbb{C}\mathrm{P}^2$'s and q copies of $\overline{\mathbb{C}\mathrm{P}^2}$'s.

Since the complement of the classes that give rise to the cones is of non-negative dimension, the number of singular points is less or equal to $\operatorname{rank} H^2(X; \mathbb{Z})$, i.e. $p + q \leq b_2(X)$. At the same time, cobordism invariance of the signature implies $p - q = \sigma(X)$. Now take Q_X to be negative-definite. Then $q - p = b_2(X) \geq p + q$ and therefore $2p \leq 0$, hence $p = 0$. Now, one needs to check that the classes $\pm c$ are mutually orthogonal to conclude that Q_X is diagonal.

6.2 Seiberg-Witten Invariants

For the rest of this section, we will assume that X is closed, connected, oriented and smooth, with $b_2^+(X) \geq 2$ (to avoid wall-crossing). Given a Spin^c structure $\mathfrak{s}$ and generic parameters $(g, \omega) \in \mathcal{P}$, $\mathcal{M}_\omega$ is a smooth, closed, oriented manifold of expected dimension (if the expected dimension is negative, $\mathcal{M}_\omega = \varnothing$). Then, the fundamental class of the moduli space is an invariant of $\mathfrak{s}$.

Theorem 6.4. *The map*

$$\begin{aligned} SW_X : \mathrm{Spin}^c(X) &\longrightarrow H_*(\mathcal{B}^*; \mathbb{Z}) \\ \mathfrak{s} &\longmapsto [\mathcal{M}_\omega] \end{aligned}$$

is an oriented diffeomorphism invariant of X. That is, if $f : Y \to X$ is an orientation-preserving diffeomorphism, then $f^* \circ SW_X = SW_Y \circ f^*$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spin}^c(X) \arrow[r, "SW_X"] \arrow[d, "f^*"', "\cong"] & H_*(\mathcal{B}^*_X; \mathbb{Z}) \arrow[d, "\cong"', "f^*"] \\ \mathrm{Spin}^c(Y) \arrow[r, "SW_Y"'] & H_*(\mathcal{B}^*_Y; \mathbb{Z}) \end{tikzcd}$$

Remark 6.5. The commutative square makes sense since $\mathcal{B}^*_X$ is a classifying space for $\mathcal{G}_X$: A homotopy equivalence $Y \to X$ induces an identification of these classifying spaces, i.e. an induced isomorphism f^*.

Proof of Theorem. We just need to check that SW_X odes not depend on the choice of parameters. Since $b_2^+(X) \geq 2$, we can connect two pairs (g, ω), (g', ω') by a path disjoint from the wall and obtain a cobordism between the moduli spaces. This shows that their homology (as a submanifold of $\mathcal{B}^*$) is the same. ☐

Remark 6.6.

(i) $[\mathcal{M}_\omega]$ depends in an uncontrollable way on the orientation of X (!).

(ii) If the moduli space has dimension 0, we obtain a *numerical invariant*.

(iii) One may work with $\mathcal{M}_\omega$ without considering its orientation and the associated unoriented invariant $SW_X : \mathrm{Spin}^c(X) \to H_*(\mathcal{B}^*; \mathbb{Z}_2)$.

6.2.1 Properties of Seiberg-Witten Invariants

We highlight some fundamental properties of SW invariants:

(i) Let $\tau : \mathrm{Spin}^c(X) \to \mathrm{Spin}^c(X)$ be the charge conjugation map. Then $SW_X(\tau(\mathfrak{s})) = \pm SW_X(\mathfrak{s})$. This is because conjugation leads to an identification $(g, \omega) \leftrightarrow (g, -\omega)$. This yields a diffeomorphism (not necessarily oriented) of $\mathcal{M}_\omega$, i.e. $\tau_*[\mathcal{M}_\omega] = \pm[\mathcal{M}_\omega]$.

(ii) SW_X has finite support.

Proof. If $SW_X(\mathfrak{s}) \neq 0$, this implies that

$$0 \leq \dim \mathcal{M}_\omega = \frac{1}{4}\left(c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X))\right)$$

so $c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X)$. Another implication of $SW_X(\mathfrak{s}) \neq 0$ is that there exist solutions for $(g, 0)$; let (A, Φ) be such a solution. Then

$$c_1^2(L_\mathfrak{s}) = \frac{1}{4\pi^2}\left(\|F^+_{\hat{A}}\|^2_{L^2} - \|F^-_{\hat{A}}\|^2_{L^2}\right)$$

where $\|F_A^+\|_{L^2}^2 \leq \|-s_g\|_{L^2}/2\sqrt{2}$. This implies a bound on $F_{\hat{A}}^-$, hence, if we write $H^2(X;\mathbb{R}) = \mathcal{H}^2(X) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$, the projection of $c_1(L)_\mathbb{R} = [iF_A/2\pi]$ to both $\mathcal{H}_\pm^2$ are contained in a bounded set (the bound depending only on (g,ω)). This implies that there are only finitely many possibilities for $c_1(L_\mathfrak{s})_\mathbb{R} \in H^2(X;\mathbb{R})$. Since we have an embedding $H^2(X;\mathbb{Z})/\operatorname{Tor} \hookrightarrow H^2(X;\mathbb{R})$ and Tor is finite, this implies that there are only finitely many $\mathfrak{s} \in \operatorname{Spin}^c(X)$ for which $SW_X(\mathfrak{s}) \neq 0$. □

(iii) If X admits a metric g_0 with $s_{g_0} > 0$, then $SW_X(\mathfrak{s}) \equiv 0$.

Proof. Suppose $SW_X(\mathfrak{s}) \neq 0$ for some $\mathfrak{s}$. Then there must be solutions of the SW equations for $(\mathfrak{s}, g_0, \omega)$ for some ω. Every solution must satisfy $|\Phi|^2 \leq \max\{0, -s_{g_0} + 2\sqrt{2}|\omega|\}$: for $\omega = 0$, and $g = g_0$, there can thus only be reducible solutions. For generic (g,ω), there are no reducible solutions and since $-s_{g_0} + 2\sqrt{2}|\omega|$ is negative at $(g_0, 0)$, it is on an open neighborhood $U \subset \mathcal{P}$ of $(g_0,0)$. Hence, for generic $(g,\omega) \in U$, the moduli space is empty: This is a contradiction. □

Remark 6.7.

(i) This fails if $b_2^+(X) \leq 1$, as exemplified by $\mathbb{C}\mathrm{P}^2$.

(ii) The above argument still works if X admits a metric g with $s_g \geq 0$ and $s_g \not\equiv 0$. This can also be shown by deforming the given metric to one with $s_g > 0$.

(iv) In case X admits a scalar-flat metric, we have the following:

Proposition 6.8. *Suppose X admits a metric g with $s_g \equiv 0$. If $2\chi(X)+3\sigma(X) \geq 0$, then $SW_X(\mathfrak{s}) = 0$ unless $c_1(L_\mathfrak{s})_\mathbb{R} = 0$ and $2\chi(X) + 3\sigma(X) = 0$. In this latter case, $SW_X(\mathfrak{s}) \in H_0(\mathcal{B}^*;\mathbb{Z})$.*

Proof. Suppose X is as in the proposition and $SW_X(\mathfrak{s}) \neq 0$ for some $\mathfrak{s}$. Then for this $\mathfrak{s}$, since $\dim \mathcal{M}_\omega \geq 0$ we have

$$c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X) \geq 0$$

On the other hand, since $SW_X(\mathfrak{s})$, there must be solutions for $(g,0)$, where g is the scalar-flat metric. Then we have $|\Phi|^2 \leq -s_g = 0 \implies \Phi \equiv 0 \implies F_A^+ = \sigma(\Phi,\Phi) \equiv 0$. But then

$$c_1^2(L_\mathfrak{s}) = -\frac{1}{4\pi^2}\int_X \|F_A^-\|^2 \mathrm{vol}_g \leq 0$$

Thus, $c_1^2(L_\mathfrak{s}) = 2\chi(X) + 3\sigma(X) = 0$. Finally, for the solutions $(A,0)$, we must have $F_A^- \equiv 0$, so A is flat on $L_\mathfrak{s} \implies c_1(L_\mathfrak{s})_\mathbb{R} = 0$, hence $c_1(L_\mathfrak{s}) \in \operatorname{Tor}$ and in the absence of torsion $c_1(L_\mathfrak{s}) = 0 \in H^2(X;\mathbb{Z})$. Since the expected dimension is proportional to $c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X)) = 0$, we see that $SW_X(\mathfrak{s}) \in H_0(\mathcal{B}^*;\mathbb{Z})$. □

Corollary 6.9. *If X is scalar flat and $2\chi(X) + 3\sigma(X) > 0$, then $SW_X \equiv 0$.*

6.3 Computations of Seiberg-Witten Invariants and Some Applications

6.3.1 $K3$ Surfaces and the 4-Torus

Let X be a smooth 4-manifold underlying a complex $K3$ surface, e.g. a smooth degree 4 hypersurface in $\mathbb{C}\mathrm{P}^3$ or the transverse intersection of three quadrics in $\mathbb{C}\mathrm{P}^5$. X is oriented by the complex structure and for this orientation $\sigma(X) = -16$, $\chi(X) = 24$. X is a *Calabi-Yau* manifold. Such a manifold admits Ricci-flat (and therefore scalar-flat) metrics, called Calabi-Yau metrics. Note that $2\chi(X) + 3\sigma(X) = 0$, so we may still have a non-trivial SW invariant. However, $\bar{X}$ has $\sigma(X) = 16$, $\chi(X) = 24$, so $2\chi(\bar{X}) + 3\sigma(\bar{X}) = 96 > 0$. By the above corollary, $SW_{\bar{X}} \equiv 0$.

The proof of proposition 6.8 shows that, since $\pi_1(K3) = 1$ and hence there is no torsion, $SW_X(\mathfrak{s}) = 0$ unless $c_1(L_\mathfrak{s}) = 0$, i.e. unless $\mathfrak{s}$ is the Spin^c structure induced by the unique Spin structure (uniqueness follows from remark 2.32). Equip X equipped with this unique Spin^c structure $\mathfrak{s}$ and consider the SW equations for $(g, 0) \in \mathcal{P}$, where g is scalar-flat. For every solution, $\Phi \equiv 0$, and A is flat: In fact A is unique up to gauge equivalence (cf. exercise 9.1, which shows that the space of flat connections is $T^{b_1(X)}$). We cannot yet conclude that $SW_X(\mathfrak{s}) = \pm 1$ since we need to perturb to a transversal situation (i.e. generic parameters).

At the reducible solution, the linearized SW equations uncouple into the half-de Rham complex

$$\Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}^+} \Omega^2_+(X)$$

which has cohomology groups $H^0_{\mathrm{hdR}} \cong \mathbb{R}$, $H^1_{\mathrm{hdR}} = 0$, $H^2_{\mathrm{hdR}} \cong \mathbb{R}^{b_2^+(X)} = \mathbb{R}^3$, and the Dirac operator. Since A is flat, the Weitzenböck formula shows that

$$D_A^- D_A^+ = \nabla_A^* \nabla_A + \frac{1}{4} s_g$$

For the Calabi-Yau metrics, $s_g \equiv 0$. Since we are interested in the kernel and cokernel, suppose that $D_A^+ \Phi = 0$. Then

$$0 = \int_X \langle D_A^- D_A^+ \Phi, \Phi \rangle \, \mathrm{vol}_g = \int_X \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle \mathrm{vol}_g = \int_X |\nabla_A \Phi|^2 \mathrm{vol}_g$$

Thus, $\ker D_A^+$ is the space of parallel, positive spinors. Similarly, $\operatorname{coker} D_A^+ = \ker D_A^-$ is the space of parallel, negative spinors. It is a general fact which we will not prove that a parallel spinor which is nonzero at some remains nonzero everywhere. However, $e(V_-) \neq 0$, hence V_- has no nowhere-vanishing section, i.e. the space of parallel, negative spinors consists only of the zero section and D_A^+ is surjective. The Atiyah-Singer index theorem shows that $\mathrm{ind}_{\mathbb{C}} D_A^+ = -\frac{1}{8}\sigma(X) = 2$. Now $\ker D_A^+ = \mathbb{C}^2$ is made up of parallel sections for ∇_A. These two parallel sections give a global trivialization of V_+, i.e. the complex

$$0 \longrightarrow \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-)$$

has cohomology $H^1_D \cong \mathbb{C}^2$ and $H^0_D = H^2_D = 0$. We conclude that the elliptic complex has cohomology

$$H^0_{(A,0)} \cong \mathbb{R} \qquad H^1_{(A,0)} \cong \mathbb{C}^2 \qquad H^2_{(A,0)} \cong \mathbb{R}^3$$

S^1 acts on $\mathbb{C}^2$ by the standard action and trivially on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$. Defining the S^1-invariant Kuranishi map

$$\begin{aligned} \psi : \mathbb{C}^2 = H^1_{(A,0)} &\longrightarrow H^2_{(A,0)} = \mathbb{C} \oplus \mathbb{R} \\ (z, w) = \Phi &\longmapsto \sigma(\Phi, \Phi) = (z\bar{w}, |z|^2 - |w|^2) \end{aligned}$$

we see that a neighbourhood of $[A, 0] \in \mathcal{M}_0$ is of the form $\psi^{-1}(0)/S^1$. Let us perturb the curvature equation by $\omega \in \mathcal{H}^2_+$ and consider $\psi^{-1}(\omega)/S^1$. ψ is the *cone on the Hopf map*, i.e. restricting to elements satisfying $|\omega|^2 = 1$ we find that $\psi^{-1}(\omega)$ is a Hopf circle in $S^3 \subset \mathbb{C}^2$, and $\psi^{-1}(\omega)/S^1$ is a point. For a generic ω, this unique solution up to gauge for the SW equation with parameters $(g, -\omega)$ is transverse, so $SW_X(\mathfrak{s}) = \pm 1$. In summary, we have proven the following:

Proposition 6.10. *If X is the smooth manifold underlying a $K3$ surface with orientation induced by the complex structure, then*

$$SW_X(\mathfrak{s}) = \begin{cases} \pm 1, & \text{if } \mathfrak{s} \text{ is induced by the unique Spin structure,} \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 6.11. *There is no orientation-preserving diffeomorphism $K3 \to \overline{K3}$. But this is already clear form the fact that $\sigma(K3) \neq 0$.*

Now, we discuss the four-torus $X = T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with a flat metric g. Since $b_2^+(T^4) = 3$, SW_{T^4} is well-defined. Since $2\chi(X) + 3\sigma(X) = 0$, $SW_{T^4}(\mathfrak{s}) = 0$ and there is no torsion, $SW_{T^4}(\mathfrak{s}) = 0$ unless $\mathfrak{s}$ is induced by a Spin structure (and in fact every Spin structure on T^4 induces the same Spin^c structure, cf. remark 2.32). For parameters $(g, 0)$ with g scalar flat, we saw that all solutions are reducible with A flat. The gauge equivalence classes of flat connections make up $H^1_{\mathrm{dR}}(X)/H^1(X;\mathbb{Z}) \cong T^4$ (cf. exercise 9.1). In this case, $\dim \mathcal{M}_0 = 4$ while the expected dimension is 0. It turns out that using the Kuranishi method for T^4 is rather cumbersome, so we use a different technique, which in fact also applies to $K3$ surfaces.

For any CCOS, Riemannian 4-manifold (X, g) which is scalar-flat with $2\chi(X) + 3\sigma(X) = 0$, consider $\mathfrak{s}$ induced by a Spin structure, i.e. $c_1(L_{\mathfrak{s}}) = 0$. Pick a parallel self-dual 2-form ω (on a Calabi-Yau manifold such as $K3$ or T^4, one may use the fundamental form) on (X, g) and consider the SW equations for $\mathfrak{s}$ with parameters (g, ω)

$$0 = \langle c_1(L_{\mathfrak{s}}) \smile [\omega], [X]\rangle = \int_X \frac{i}{2\pi} F_A \wedge \omega = \int_X \frac{i}{2\pi} F_A^+ \wedge \omega$$

where we used self-duality of ω. Let $(A, \Phi) \in \mathcal{Z}_\omega$. Then the above equation, combined with the Weitzenböck formula, yields:

$$\begin{aligned} 0 &= \int_X \langle D_A^- D_A^+ \Phi, \Phi\rangle \mathrm{vol}_g = \int_X \Big(|\nabla^A \Phi|^2 + \frac{1}{2}\langle \gamma(F_{\hat{A}}^+)\Phi, \Phi\rangle\Big)\mathrm{vol}_g \\ &= \int_X \Big(|\nabla^A \Phi|^2 + 2\langle F_{\hat{A}}^+, \sigma(\Phi, \Phi)\rangle\Big)\mathrm{vol}_g = \int_X \Big(|\nabla^A \Phi|^2 + 2\langle F_{\hat{A}}^+, F_{\hat{A}}^+ - \omega\rangle\Big)\mathrm{vol}_g \\ &= \int_X \Big(|\nabla^A \Phi|^2 + 2|F_{\hat{A}}^+|^2\Big)\mathrm{vol}_g \end{aligned}$$

Hence, Φ is parallel and $F_{\hat{A}}^+ \equiv 0$. Since $c_1^2(L_{\mathfrak{s}})_{\mathbb{R}} = 0$, we conclude that $F_{\hat{A}}^- = 0$ as well, so A is flat. Notice that $\Phi \not\equiv 0$ since $\sigma(\Phi, \Phi) = -\omega$ by the curvature equation. Another linearly independent parallel spinor is given by $J(\Phi)$, where J is the charge conjugation map. Since V_+ admits a trivialization of V_+ by parallel sections, A is a product connection: Such connections are unique up to gauge. The map

$$\begin{aligned} \mathbb{C}^2 = \ker D_A^+ &\longrightarrow \mathcal{H}^2_+ \\ \Phi &\longmapsto \sigma(\Phi, \Phi) \end{aligned}$$

has $\sigma^{-1}(-\omega) \cong S^1$ with the constant gauge transformations acting freely on this circle, since $\sigma(\lambda\Phi, \lambda\Phi) = \sigma(\Phi, \Phi)$ if and only if $\lambda \in S^1$. Hence, the solution (A, Φ) is unique up to gauge. Replacing ω by $r\omega$ for some generic $r \in \mathbb{R}$, we can make $f_{r\omega}$ transverse. We have therefore shown the following.

Proposition 6.12. *The SW invariant for the torus T^4 is given by*

$$SW_{T^4}(\mathfrak{s}) = \begin{cases} \pm 1, & \text{for } \mathrm{Spin}^c \text{ structures induced by } \mathrm{Spin} \text{ structures,} \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 6.13. *$K3$ and T^n, for $n \leq 4$, do not admit metrics with positive scalar curvature.*

Proof. For $K3$ or T^4 this follows from $SW_X \not\equiv 0$. For T^n with $n < 4$, the claim follows by taking products, since a product of a positive scalar curvature metric with a flat metric has positive scalar curvature. □

Remark 6.14. For $K3$, this already from the fact that $K3$ is Spin and has nonzero signature; see 4.16. In the case of T^2, the corollary follows from the Gauss-Bonnet theorem:

$$0 = \chi(T^2) = \frac{1}{2\pi}\int_{T^2} K \mathrm{vol}_g$$

6.3.2 Einstein Manifolds

Definition 6.15 (Einstein Metric)**.** A Riemannian metric is called Einstein if $\mathrm{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$ or, equivalently, if the trace-free part of the Ricci tensor, $\mathrm{Ric}_{g,0}$, vanishes.

By taking the trace and using that the trace of g is constant, we see that an Einstein metric always has constant scalar curvature. The following are some first examples of Einstein manifolds.

Example 6.16.

- Spaces of constant curvature: S^4/Γ, $\mathbb{R}^4/\Gamma \cong T^4/\Gamma'$, $\mathbb{H}^4/\Gamma$, and so on, where Γ, Γ' are discrete groups acting freely (and properly discontinuously) by isometries.
- Other locally symmetric spaces: $(\mathbb{CP}^2, \omega_{\text{FS}})$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CH}^2/\Gamma$, $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$.
- Calabi-Yau metrics on $K3$ and finite quotients $K3/\Gamma$.
- Certain Kähler-Einstein metrics with $s_g > 0$ on $\mathbb{CP}^2 \# k\overline{CP}^2$, for $3 \leq k \leq 8$.
- Kähler-Einstein metrics with $s_g < 0$ on compact complex surfaces with ample canonical bundle. This is a large class of manifolds, which we will soon discuss.
- $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with the so-called Page metric.

We use the following result without proof:

Theorem 6.17 (Chern-Gauss-Bonnet in Dimension 4)**.** *The Euler characteristic of a compact, oriented 4-manifold X is given by*

$$\chi(X) = \frac{1}{8\pi^2} \int_X \left(\frac{1}{24} s_g^2 + \|W_+\|^2 + \|W_-\|^2 - \| \operatorname{Ric}_{g,0} \|^2 \right) \operatorname{vol}_g$$

where $W = W_+ + W_-$ is the Weyl tensor.

As a corollary, we have:

Proposition 6.18 (Berger)**.** *If a 4-manifold X is Einstein, then $\chi(X) \geq 0$ with equality only if the Einstein metric is flat.*

We see that $S^1 \times S^3$ and $T^4 \# T^4$ do not admit Einstein metrics.

Proposition 6.19 (Thorpe)**.** *If X^4 is Einstein, $\chi(X) \geq 3|\sigma(X)|/2$.*

Proof. For this, we need a curvature expression for the first Pontryagin number:

$$\sigma(X) = \frac{1}{3}\langle p_1(X), [X] \rangle = \frac{1}{12\pi^2} \int_X \left(|W_+|^2 - |W_-|^2 \right) \operatorname{vol}_g$$

The Chern-Gauss-Bonnet theorem then yields

$$\begin{aligned} 2\chi(X) &= \frac{1}{4\pi^2} \int_X \left(\frac{1}{24} s_g^2 + |W_+|^2 + |W_-|^2 \right) \operatorname{vol}_g \geq \left| \frac{1}{4\pi^2} \int_X \left(|W_+|^2 - |W_-|^2 \right) \operatorname{vol}_g \right| \\ &= 3|\sigma(X)| \end{aligned}$$

□

Remark 6.20. Something more can be said: In the case of equality, the Einstein metric must have $s_g \equiv 0$ and either $W_+ \equiv 0$ or $W_- \equiv 0$. Picking an orientation, we may assume $W_+ \equiv 0$, i.e. $2\chi(X) = -3\sigma(X)$. If W_- *also* vanishes, X is a flat and therefore, by a theorem of Bieberbach, a quotient of T^4. If not, it is locally Calabi-Yau and locally isometric to $K3$, i.e. X is always a quotient of either T^4 or $K3$ by a finite group acting freely by isometries. This observation is due to Hitchin.

Corollary 6.21. *If X^4 is* Spin *and admits an Einstein metric, then $b_2(X) \geq \frac{11}{8}|\sigma(X)|$.*

Proof. Assume $\sigma(X) \neq 0$; by Rohlin's theorem, $|\sigma(X)| \geq 16$. We compute:

$$\begin{aligned} 8b_2(X) &= 8(\chi(X) - 2 + 2b_1(X)) \\ &\geq 8 \cdot \frac{3}{2}|\sigma(X)| - 16 = 12|\sigma(X)| - 16 = 11|\sigma(X)| + (|\sigma(X)| - 16) \\ &\geq 11|\sigma(X)| \end{aligned}$$

□

Finally, we can say something about the SW invariants of Einstein manifolds:

Proposition 6.22. *Suppose X is a CCOS 4-manifold with $b_2^+(X) \geq 2$. If X admits a Ricci-flat metric and is not $K3$ or T^4, then $SW_X \equiv 0$.*

Proof. Hitchin's characterization of Einstein manifolds with $2\chi(X) = 3|\sigma(X)|$ shows that such manifolds are (quotients of) $K3$ or T^4. One can explicitly check that the nontrivial quotients of $K3$ and T^4 do not satisfy the above assumptions. Therefore, we only have to consider $2\chi(X) > 3|\sigma(X)|$. Thus $2\chi(X) + 3\sigma(X) \geq 2\chi(X) - 3|\sigma(X)| > 0$. This, together with the fact that X is scalar-flat, yields $SW_X \equiv 0$ by corollary 6.9. □

We can compare the results of Thorpe and Hitchin with the following:

Theorem 6.23 (Le Brun)**.** *Assume X is closed and oriented with $b_2^+(X) \geq 2$ and $SW_X \neq 0$. If (X, g) is Einstein, then $\chi(X) \geq 3\sigma(X)$, with equality only if g is flat, or $X = \mathbb{C}H^2/\Gamma$ (a quotient of the complex hyperbolic ball) and g is a rescaling of the standard (Bergman) metric.*

Proof. Let g be the Einstein metric and $\mathfrak{s}$ a Spin^c structure such that $SW_X(\mathfrak{s}) \neq 0$. Then there must be solutions (A, Φ) to the SW equations for parameters $(g, 0)$. Since $\dim^{\exp}\mathcal{M}_\omega \geq 0$, $c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X)$. At the same time, Chern-Weil theory tells us that

$$\begin{aligned} c_1^2(L_\mathfrak{s}) &= \frac{1}{4\pi^2}\int_X (|F_{\hat{A}}^+|^2 - |F_{\hat{A}}^-|^2)\mathrm{vol}_g \\ &\leq \frac{1}{4\pi^2}|F_{\hat{A}}^+|^2\mathrm{vol}_g = \frac{1}{32\pi^2}\int_X |\Phi|^4\mathrm{vol}_g \\ &\leq \frac{1}{32\pi^2}\int_X s_g^2\mathrm{vol}_g \leq 3\left(\frac{1}{4\pi^2}\int_X \left(\frac{1}{24}s_g^2 + 2|W_-|^2\right)\mathrm{vol}_g\right) = 3(2\chi(X) - 3\sigma(X)) \end{aligned}$$

In conclusion, we find:

$$6\chi(X) - 9\sigma(X) \geq 2\chi(X) + 3\sigma(X) \iff \chi(X) \geq \sigma(X)$$

If we have equality, we see that $\dim\mathcal{M}_0 = 0$, $F_{\hat{A}}^- \equiv 0$, $|\Phi|^2 = -s_g$ and $W_- \equiv 0$. Using corollary 4.24, we also see that $\nabla^A\Phi \equiv 0$. Therefore, the curvature equation $F_{\hat{A}}^+ = \sigma(\Phi, \Phi)$ implies that $F_{\hat{A}}^+$ is also parallel.

Since for $s_g > 0$ one always has $SW_X \equiv 0$, we may assume $s_g \leq 0$. In case $s_g \equiv 0$, we see that $\Phi \equiv 0$, hence $F_{\hat{A}}^+ \equiv 0$. In this case, the Chern-Gauss-Bonnet theorem shows

$$\chi(X) = \frac{1}{8\pi^2}\int_X |W_+|^2\mathrm{vol}_g \qquad\qquad \sigma(X) = \frac{1}{12\pi^2}\int_X |W_+|^2\mathrm{vol}_g$$

But then$\chi(X) = \frac{3}{2}\sigma(X)$, but at the same time $\chi(X) = 3\sigma(X)$, hence $\chi(X) = \sigma(X) = 0$. Then $W_+ \equiv 0$, hence all components of the Riemann tensor vanish and (X, g) is flat.

Now assume $s_g < 0$. Then $\Phi \not\equiv 0$ and $F_{\hat{A}}^+$ is a parallel, self-dual 2-form, which is up to scaling a Kähler form for g. But then g is a Kähler metric (to be discussed in the next section), i.e. (X, g) is Kähler-Einstein with $s_g < 0$ and $W_- \equiv 0$. In this situation, it is a general fact that W_+ is parallel. Therefore, the Riemann tensor is parallel and therefore this manifold is a *locally symmetric space*, which means that its universal Riemannian cover is a symmetric space. The universal Riemannian cover is a non-compact, Hermitian symmetric space and by the classification of symmetric spaces, it must be either $\mathbb{C}H^2$ or $H^2 \times H^2$. Since $\chi(X) = 3\sigma(X) > 0$ by Chern-Gauss-Bonnet, we can rule out $H^2 \times H^2$. □

Remark 6.24.

(i) In the last step of this proof, we make use of Hirzebruch's *proportionality principle*, which asserts that (certain) characteristic numbers of locally symmetric spaces like the above are proportional to those of a certain "compact dual" space which is naturally associated to them. This correspondence associates $\mathbb{C}\mathrm{P}^2$ to $\mathbb{C}H^2$ and its quotients, which therefore always satisfy $\chi = 3\sigma$. Similarly, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ corresponds to $H^2 \times H^2$ and its quotients, which therefore have $\chi > 0$, $\sigma = 0$.

(ii) Comparing this with the results by Hitchin-Thorpe, we see that if the manifold is oriented such that $\sigma(X) \geq 0$, the non-vanishing of SW invariants imposes a stronger topological constraint than the Einstein assumption alone. However, since the SW invariants depend strongly on the orientation (requiring their non-triviality typically *fixes* the orientation), the result by Le Brun sometimes yields no extra information over Hitchin-Thorpe.

6.3.3 Complex Surfaces with Kähler-Einstein Metrics

In this section, we will describe the SW invariants of one of the most important classes of Einstein manifolds: Compact complex surfaces equipped with Kähler-Einstein metrics with negative scalar curvature and ample canonical bundle. We first introduce some background material: Suppose that J is an almost complex structure on a 4-manifold X, i.e. $J \in \Gamma(\mathrm{End}(TX))$ with $J^2 = -\mathrm{Id}$, which gives TX the structure of a complex vector bundle. Recall that, in this context, the first Pontryagin class of X is given by

$$p_1(X) = c_1^2(X) - 2c_2(X) \implies c_1^2(X) = p_1(X) + 2c_2(X)$$

Then by the signature formula 3.44 and the fact that $c_2(X) = e(X)$ since (TX, J) is of complex rank two, we see that $\langle c_1^2(X), [X]\rangle = 3\sigma(X) + 2\chi(X)$. Since every integral lift of $w_2(X)$ (such as $c_1(x)$) induces a Spin^c-structure, every J must give rise to a Spin^c structure. In conclusion, almost complex manifolds come with a canonical Spin^c structure, $\mathfrak{s}_{\text{can}}$.

We extend J $\mathbb{C}$-linearly to $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$, where $T^{1,0}$ ($T^{0,1}$) is the eigenspace of J with eigenvalue $+i$ ($-i$). As a complex vector bundle, $T^{1,0} \cong (TX, J)$ while $T^{0,1} \cong (TX, -J)$. We get an analogous decomposition of $T^*X \otimes_{\mathbb{R}} \mathbb{C}$ and its exterior powers:

$$\Lambda^k(T^*X \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=r} \left(\Lambda^p(T^{1,0})^* \otimes \Lambda^q(T^{0,1})^*\right) = \bigoplus_{\substack{p+q=k \\ p,q\geq 0}} \Lambda^{p,q}$$

where $\Lambda^{1,0} = (T^{1,0})^*$.

Definition 6.25 (Hermitian Metric)**.** A metric on TX (and the related bundles) with respect to which J is orthogonal, i.e. $g(Jv, Jw) = g(v, w)$, is called a Hermitian metric for J.

Given (X, g, J) where g and J are compatible (i.e. g is Hermitian), the two-form ω defined by $\omega(X, Y) = g(JX, Y)$ is called the *fundamental form*. ω is is non-degenerate and prescribes an orientation for X, since its top power is non-vanishing and therefore defines a volume form.

Definition 6.26 (Kähler Manifold)**.** The triple (J, g, ω) makes X into a Kähler manifold if $\nabla J = 0$, or equivalently $\nabla\omega = 0$. If $\mathrm{d}\omega = 0$, we call X *almost* Kähler.

Note that $\nabla J = 0$ implies that J is integrable, while $\nabla\omega = 0$ implies that ω is closed. Thus, Kähler manifolds are almost Kähler and come with both a complex and symplectic structure, which are mutually compatible.

Given a Hermitian metric g, we extend the Hodge star operator to the complexified differential forms $* : \Lambda^{p,q} \to \Lambda^{2-q,2-p}$, given by the formula $\alpha \wedge *\bar{\beta} = g(\alpha, \beta)\mathrm{vol}_g$, which makes $*$ into a $\mathbb{C}$-linear map. This allows us to consider the spaces of self-dual and anti self-dual forms:

$$\Lambda^2_+ \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\,\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}$$
$$\Lambda^2_- \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,1}_{\perp_\omega}$$

where we used that $\omega \in \Lambda^{1,1}$ to define the orthogonal complement $\Lambda^{1,1}_{\perp_\omega} \subset \Lambda^{1,1}$. Now, we explicitly define the canonical Spin^c structure defined by the almost complex structure. Its spinor bundles are given by

$$V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \qquad\qquad V_- = \Lambda^{0,1}$$

and we define Clifford multiplication by

$$\begin{aligned} \gamma(a) : V_+ &\longrightarrow V_- \\ (\alpha, \beta) &\longmapsto \sqrt{2}(a^{0,1} \wedge \alpha - *(a^{1,0} \wedge *\beta)) \end{aligned}$$

and

$$\begin{aligned} \gamma(a) : V_- &\longrightarrow V_+ \\ \psi &\longmapsto \sqrt{2}\left(- * (a^{1,0} \wedge *\psi), a^{0,1} \wedge \psi\right) \end{aligned}$$

One has to check that this definition satisfies the properties of a Clifford module; this is done in exercise 12.1. Let $\alpha \in \Omega^{p,q}(X)$ and define the Dolbeault operators $\partial\alpha := (\mathrm{d}\alpha)^{p+1,q}$ and $\bar{\partial}\alpha := (\mathrm{d}\alpha)^{p,q+1}$. Keep in mind that in the almost complex case, we do not have $\mathrm{d} = \partial + \bar{\partial}$ in general. We can still define the adjoint operators with respect to the L^2 metric. Denote them by ∂^* and $\bar{\partial}^*$. Then our definition for Clifford multiplication makes γ into the symbol of the maps

$$\begin{aligned} V_+ &\longrightarrow V_- \\ (\alpha, \beta) &\longmapsto \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) \end{aligned}$$

$$\begin{aligned} V_- &\longrightarrow V_+ \\ \psi &\longmapsto \sqrt{2}(-\bar{\partial}^*\psi, -\bar{\partial}\psi) \end{aligned}$$

This canonical Spin^c structure $\mathfrak{s}_{\text{can}}$ gives us a way to identify $\mathrm{Spin}^c(X)$ with $H^2(X;\mathbb{Z})$ via the map $H^2(X;\mathbb{Z}) \ni L \mapsto \mathfrak{s}_{\text{can}} \otimes E =: \mathfrak{s}_E$, where we abuse notation to identify a line bundle E with its first Chern class (we will sometimes also "additive" notation in place of tensor products, e.g. $K - E$ may denote the bundle $K \otimes E^{-1}$). The spinor bundles of $\mathfrak{s}_E$ are given by $V_+ = E \oplus (\Lambda^{0,2} \otimes E) = E \oplus (K^{-1} \otimes E)$, where $K := \Lambda^{n,0}$ is the canonical line bundle, and $V_- = \Lambda^{0,1} \otimes E$. However, this choice of "reference point"' is not an oriented diffeomorphism invariant, thus the identification is not natural under pullback.

Lemma 6.27. *If (J, g, ω) is Kähler-Einstein, then*

$$\langle c_1^2(X, J), [X] \rangle = \frac{1}{32\pi^2} \int_X s_g^2 \mathrm{vol}_g$$

Proof. Let ρ be the curvature 2-form of the connection on the characteristic line bundle $\det V_\pm = K^{-1}$ induced by the Levi-Cività connection. We will need the following fact: If g is Kähler Einstein, then $\rho = i\lambda\omega$ for constant λ. In the present case, $\lambda = s_g/4$ (this can be computed from Chern-Gauss-Bonnet). Since $c_1(X, J) = -c_1(K)$, we have

$$\begin{aligned} \langle c_1^2(X, J), [X] \rangle = \langle c_1^2(K), [X] \rangle &= \int_X \frac{i}{2\pi}\rho \wedge \frac{i}{2\pi}\rho \\ &= \int_X \left(\frac{-s_g}{8\pi}\omega\right) \wedge \left(\frac{-s_g}{8\pi}\omega\right) = \frac{1}{64\pi^2} \int_X s_g^2 \omega \wedge \omega \end{aligned}$$

Finally, note that $\omega^2 = 2\mathrm{vol}_g$ to obtain the promised result. $\square$

Assume X has a Kähler-Einstein structure (J, g, ω) and E is a complex line bundle such that $SW_X(\mathfrak{s}_E) \neq 0$. Then the SW equations for $\mathfrak{s}_E$ and $(g, 0) \in \mathcal{P}$ must have solutions (A, Φ). We have

$$\begin{aligned}\langle c_1^2(X,J),[X]\rangle = 2\chi(X)+3\sigma(X) \le \langle c_1^2(L_{\mathfrak{s}_E}),[X]\rangle &= \frac{1}{4\pi^2}\int_X \left(|F_A^+|^2 - |F_A^-|^2\right)\mathrm{vol}_g \\ \le \frac{1}{4\pi^2}\int_X |F_A^+|^2 \mathrm{vol}_g = \frac{1}{32\pi^2}\int_X |\Phi|^4 \mathrm{vol}_g &\le \frac{1}{32\pi^2}\int_X s_g^2 \mathrm{vol}_g = \langle c_1^2(X,J),[X]\rangle\end{aligned}$$

Hence, all the inequalities are equations and as a consequence $\dim \mathcal{M}_0 = 0$, $F_{\hat{A}}^- \equiv 0$ and $|\Phi|^4 = s_g^2 = \text{const}$. We have the following cases.

- If $s_g > 0$, and $b_2^+(X) \ge 2$, then $SW_X \equiv 0$. In fact, this case does not occur since it turns out that $b_2^+(X) = 1$ for a Kähler-Einstein surface X.
- If $s_g \equiv 0$, then g is flat or Calabi-Yau, i.e. either T^4 or $K3$.
- For $s_g < 0$, we see that $|\Phi|^2 = -s_g > 0$, so the solution is irreducible. Recall from corollary 4.24 that if the Spin^c structure $\mathfrak{s}$ admits solutions for parameters $(g, 0) \in \mathcal{P}$, then

  $$\langle c_1^2(L_{\mathfrak{s}}),[X]\rangle \le \frac{1}{32\pi^2}\int_X s_g^2 \mathrm{vol}_g$$

  with equality if and only if $F_A^- = 0$, $\nabla_A \Phi = 0$, and $|\Phi|^2 = -s_g$. Hence, Φ is non-zero and parallel; it is a fact that such a section trivializes either E or $K^{-1} \otimes E$. In the first case, $\mathfrak{s}_E = \mathfrak{s}_{\text{can}}$. In the second case, $\mathfrak{s}_E = \mathfrak{s}_{\text{can}} \otimes K = \overline{\mathfrak{s}_{\text{can}}}$, since K is the characteristic line bundle. These are the only Spin^c-structures for which $SW_X(\mathfrak{s}_E)$ may be non-zero. In both cases, there are in fact tautological solutions, unique up to gauge.

The above discussion can be summarized as follows:

Theorem 6.28 (Witten)**.** *If X is a complex surface with $b_2^+(X) \ge 2$, which admits a Kähler-Einstein metric with $s_g < 0$, then*

$$SW_X(\mathfrak{s}) = \begin{cases} \pm 1 & \textit{if } \mathfrak{s} = \mathfrak{s}_{can} \textit{ or } \bar{\mathfrak{s}}_{can} \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 6.29. In the situation of the theorem above, suppose that $f : X \to X$ is an orientation-preserving diffeomorphism. Then we must have $f^*(\mathfrak{s}_{\text{can}}) = \mathfrak{s}_{\text{can}}$. For the action of f^* on $H^2(X;\mathbb{Z})$, this implies that $f^* c_1(X,J) = \pm c_1(X,J) \neq 0$, or equivalently $f^*K = \pm K$ (denoting the first Chern class of the canonical bundle by K).

Our computations so far can be summarized as follows:

Theorem 6.30. *Let X be one of the following Kähler 4-manifolds*

- *$X = T^4$.*
- *$X = K3$.*
- *X Kähler-Einstein with $s_g < 0$ and $b_2^+ \ge 2$.*

Then $SW_X(\mathfrak{s}) = \pm 1$ if $\mathfrak{s} = \mathfrak{s}_{can}$ or $\mathfrak{s} = \bar{\mathfrak{s}}_{can}$, and $SW_X(\mathfrak{s}) = 0$ otherwise.

6.3.4 The Adjunction Inequality for Surfaces

In an earlier chapter, we saw (cf. theorem 3.40):

Theorem 6.31. *If X is a closed, complex 4-manifold and Σ an embedded complex curve equipped with the complex orientation, then $2g(\Sigma) - 2 = \Sigma \cdot \Sigma + K\Sigma$.*

In the presence of non-vanishing SW invariants, we obtain a lower bound for any smoothly embedded surface:

Theorem 6.32 (Adjunction Inequality for Surfaces)**.** *Let X be a CCOS 4-manifold with $b_2^+(X) \geq 2$, and $\Sigma \subset X$ a smoothly embedded, oriented (and connected) surface with $g(\Sigma) \neq 0$ and $\Sigma \cdot \Sigma \geq 0$. If $SW_X(\mathfrak{s}) \neq 0$, then*

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |c_1(L_\mathfrak{s}) \cdot \Sigma|$$

This inequality gives us a lower bound on the genus $g(\Sigma)$ if $[\Sigma]$ is fixed with positive self-intersection.

Corollary 6.33. *Let X be a Kähler 4-manifold as in theorem 6.30. Suppose $\Sigma \subset X$ is a complex curve with $\Sigma \cdot \Sigma \geq 0$ and let Σ' be a connected, smoothly embedded surface with $[\Sigma'] = [\Sigma]$. Then $g(\Sigma') \geq g(\Sigma)$.*

Proof. Since X is Kähler and $\Sigma \subset X$ complex, Σ is also Kähler and hence $\langle \omega, [\Sigma] \rangle \neq 0$, i.e. $[\Sigma] \neq 0$ and is of infinite order. Now if Σ' is a sphere such that $g(\Sigma') = g(\Sigma)$ and $\Sigma' \cdot \Sigma' \geq 0$, then by exercises 11.1-3 we must have $SW_X \equiv 0$, but that is false by assumption. Hence $g(\Sigma') > 0$. Now, we use the adjunction inequality for $\mathfrak{s}_{\text{can}}$, since we know that $SW_X(\mathfrak{s}_{\text{can}}) \neq 0$. This yields

$$2g(\Sigma') - 2 \geq \Sigma \cdot \Sigma + |K \cdot \Sigma| \geq \Sigma \cdot \Sigma + K\Sigma = 2g(\Sigma) - 2$$

and therefore $g(\Sigma') \geq g(\Sigma)$. ◻

Remark 6.34.

(i) The Thom conjecture asks whether complex curves with $\Sigma \cdot \Sigma \geq 0$ in a complex surface have minimal genus among all smooth surfaces in their homology class. The above is a special case. The Thom conjecture was proven in 1994, very soon after the advent of SW theory, by Kronheimer and Mrowka.

(ii) The result actually holds for all symplectic 4-manifold, with the same adjunction formula, and the notion of "complex" replaced with "J-holomorphic" (J is an ω-compatible almost complex structure). It is proven using $c_1(L_{\mathfrak{s}_{\text{can}}}) = -K$. Other generalizations, such as the *symplectic* Thom conjecture (now a theorem, due to Ozsváth and Szabó), are also of interest.

We start by considering the case $\Sigma \cdot \Sigma = 0$:

Theorem 6.35. *Let X be a closed, oriented, smooth 4-manifold with $b_2^+ \geq 2$. Let $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $SW_X(\mathfrak{s}) \neq 0$. If $\Sigma \subset X$ is a smoothly embedded connected surface with $g(\Sigma) \neq 0$ and $\Sigma \cdot \Sigma = 0$, then*

$$2g(\Sigma) - 2 \geq |c_1(L_\mathfrak{s}) \cdot \Sigma|$$

Definition 6.36 (Basic Class)**.** If $SW_X(\mathfrak{s}) \neq 0$, then $c_1(L_\mathfrak{s})$ is called a basic class.

Remark 6.37. The adjunction inequality means that basic classes of smooth 4-manifolds generalize canonical classes of Kähler or symplectic manifolds.

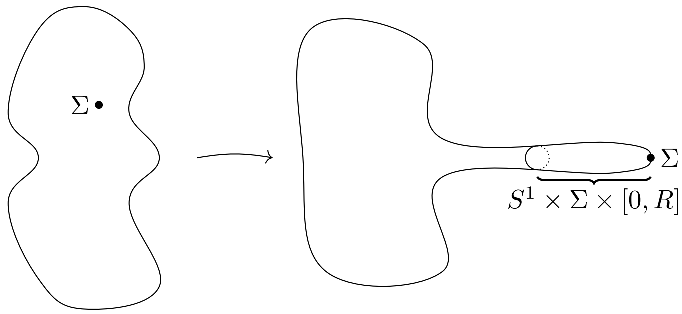
Proof of Theorem. Let g be a Riemannian metric on X and $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $SW_X(\mathfrak{s}) \neq 0$. There must exist solutions for parameters $(g, 0)$. Observe that

$$\frac{1}{4\pi^2} \int_X |F_{\hat{A}}|^2 \mathrm{vol}_g = \frac{1}{2\pi^2} \int_X |F_{\hat{A}}^+|^2 \mathrm{vol}_g - c_1^2(L_\mathfrak{s})$$

and by the monopole equations this becomes

$$\frac{1}{4\pi^2} \int_X |F_{\hat{A}}|^2 \mathrm{vol}_g = \frac{1}{2\pi^2} \int_X |\sigma(\Phi, \Phi)|^2 \mathrm{vol}_g - c_1^2(L_\mathfrak{s}) = \frac{1}{16\pi^2} \int_X |\Phi|^4 \mathrm{vol}_g - c_1^2(L_\mathfrak{s}) \leq \frac{1}{16\pi^2} \int_X s_g^2 - c_1^2(L_\mathfrak{s})$$

Choose a metric g_Σ on the embedded surface Σ which is of constant curvature, and such that $\mathrm{vol}_{g_\Sigma}(\Sigma) = 1$. Since $\Sigma \cdot \Sigma = 0$, the normal bundle $\nu(\Sigma)$ is trivial, hence diffeomorphic to $\Sigma \times D^2$. Choose a metric on X such that in $\nu(\Sigma)$ there is a metric cylinder of the form $\Sigma \times S^1 \times [0, R]$ equipped with product metric, given by g_Σ times the standard metrics on S^1 and $[0, R] \subset \mathbb{R}$; additionally, on $X \setminus \nu(\Sigma)$ the metric should not depend on R. Call it g_R.



There exist solutions to the SW equations for parameters $(g_R, 0)$ for any $R > 0$. The curvature satisfies

$$\begin{aligned}\frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 \mathrm{vol}_{g_R} &\geq \int_{\Sigma \times S^1 \times [0,R]} \left|\frac{i}{2\pi}F_{\hat{A}}\right|^2 \mathrm{vol}_{g_R} = \int_{S^1 \times [0,R]} \int_\Sigma \left|\frac{i}{2\pi}F_{\hat{A}}\right|^2 \mathrm{vol}_\Sigma \wedge \mathrm{vol}_{S^1 \times [0,R]} \\ &\geq \int_{S^1 \times [0,R]} \left(\int_\Sigma \frac{i}{2\pi}F_{\hat{A}}\right)^2 \mathrm{vol}_{S^1 \times [0,R]} = \int_{S^1 \times [0,R]} (c_1(L_\mathfrak{s}) \cdot \Sigma)^2 \mathrm{vol}_{S^1 \times [0,R]} = R(c_1(L_\mathfrak{s}) \cdot \Sigma)^2\end{aligned}$$

where we discarded part of our integral in the first step, then applied Fubini's theorem and later used the fact that we set $\mathrm{vol}_{S^1} = 1$. To relate this to our earlier inequality in an interesting way, we consider the scalar curvature term:

$$\begin{aligned}\int_X s_{g_R}^2 \mathrm{vol}_{g_R} &= \int_{\Sigma \times S^1 \times [0,R]} s_{g_R}^2 \mathrm{vol}_{g_R} + C = R\int_\Sigma s_{g_\Sigma}^2 \mathrm{vol}_{g_\Sigma} + C = R\left(\int_\Sigma s_{g_\Sigma} \mathrm{vol}_{g_\Sigma}\right)^2 + C \\ &= R(4\pi(2 - 2g(\Sigma)))^2 + C\end{aligned}$$

Here, C is a constant (i.e. independent of R). Since s_{g_Σ} is constant, we may the square out of the integral, as we did. Then we recall that on a surface, s_g is twice the Gauss curvature and that its integral therefore equals $4\pi\chi(\Sigma)$. This shows that for any $R > 0$

$$R(c_1(L_\mathfrak{s}) \cdot \Sigma)^2 \leq \frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 \mathrm{vol}_{g_R} \leq \frac{1}{16\pi^2} s_{g_R}^2 \mathrm{vol}_{g_R} - c_1^2(L_\mathfrak{s}) = R(2g(\Sigma) - 2)^2 + C'$$

where the constant C' contains anything that does not depend on R. Since $R > 0$, we have

$$(c_1(L_\mathfrak{s}) \cdot \Sigma)^2 \leq (2g(\Sigma) - 2)^2 + \frac{C'}{R}$$

and in the limit $R \to \infty$, we can use the assumption $g(\Sigma) > 0$ to obtain

$$|c_1(L_\mathfrak{s}) \cdot \Sigma| \leq 2g(\Sigma) - 2$$

which is the result we were after. $\square$

For the case $\Sigma \cdot \Sigma > 0$, we only give a sketch of proof:

Sketch of Proof of the Adjunction Inequality. Suppose $\Sigma \cdot \Sigma = k > 0$. Since $c_1(L_{\bar{\mathfrak{s}}}) = -c_1(L_\mathfrak{s})$ and $SW_X(\bar{\mathfrak{s}}) = \pm SW(\mathfrak{s})$, we may assume that $c_1(L_\mathfrak{s}) \cdot \Sigma \geq 0$. Instead of $\Sigma \subset X$, consider $\Sigma' = \Sigma \# \mathbb{C}\mathrm{P}^1 \subset X' = X \# \overline{\mathbb{C}\mathrm{P}^2}$. Since Σ' arises from tubing together Σ and a sphere, its genus is the same, however $\Sigma' \cdot \Sigma' = k - 1$ and $Q_{X'} = Q_X \oplus (-1)$. If we can find a Spin^c structure $\mathfrak{s}'$ on X' such that $SW_X(\mathfrak{s}') \neq 0$ and $c_1(L_{\mathfrak{s}'}) \cdot \Sigma' = c_1(L_\mathfrak{s}) \cdot \Sigma + 1$, then all of our assumptions still hold and we do not spoil the (prospective) adjunction inequality. Thus, given such a Spin^c structure we can inductively reduce the proof to the case $k = 0$.

Consider a linear $\mathbb{C}\mathrm{P}^1 \subset \overline{\mathbb{C}\mathrm{P}^2}$ and its Poincaré dual $x \in H^2(X'; \mathbb{Z})$. If we can find $\mathfrak{s}'$ such that $c_1(L_{\mathfrak{s}'}) = c_1(L_{\mathfrak{s}}) - x$, then we can use the fact that x is dual to a surface inside $\overline{\mathbb{C}\mathrm{P}^2}$ (and therefore intersects Σ trivially) and has self-intersection -1 to find $c_1(L_{\mathfrak{s}'}) \cdot \Sigma' = c_1(L_{\mathfrak{s}}) \cdot \Sigma + 1$, hence such a Spin^c structure would suffice for our purposes. There exists such an $\mathfrak{s}'$ if and only if the modulo two reduction of $c_1(L_{\mathfrak{s}'})$ equals $w_2(X') = w_2(X) + w_2(\overline{\mathbb{C}\mathrm{P}^2})$. We know that $c_1(L_{\mathfrak{s}})$ reduces to $w_2(X)$ while $-x$ reduces to $w_2(\overline{\mathbb{C}\mathrm{P}^2})$, hence such a Spin^c structure exists. The *blow-up formula* $SW_{X'}(\mathfrak{s}') = \pm SW_X(\mathfrak{s})$, which we will not prove, ensures that the SW invariant of $\mathfrak{s}'$ is non-trivial. This finishes the proof. $\square$

6.4 Seiberg-Witten Invariants of Symplectic Manifolds

Let (X, ω) be a CCOS, symplectic 4-manifold, i.e. $\omega \wedge \omega > 0$, and $\mathrm{d}\omega = 0$, equipped with an almost complex structure J. As we saw in the previous section, this determines a canonical Spin^c-structure $\mathfrak{s}_{\text{can}}$.

Definition 6.38 (Canonical Class)**.** The canonical class $K = K_\omega$ is the first Chern class of the complex line bundle $\Lambda^{2,0} \to X$ for any compatible almost complex structure. This is independent of the choice of almost complex structure since the space of compatible almost complex structures is contractible.

Recall the bijection $H^2(X; \mathbb{Z}) \to \mathrm{Spin}^c(X)$ which sends $E \mapsto \mathfrak{s}_{\text{can}} \otimes E =: \mathfrak{s}_E$.

Definition 6.39. We set

$$SW_{X,\omega}(E) = SW_X(\mathfrak{s}_{\text{can}} \otimes E)$$

where the subscript ω indicates the dependence on ω, since $\mathfrak{s}_{\text{can}}$ depends on ω.

To see how this works, recall that $\mathfrak{s}_{\text{can}}$ has $V_+ = \underline{\mathbb{C}} \oplus K^{-1}$ and $L_{\mathfrak{s}_{\text{can}}} = K^{-1}$, thus $\bar{\mathfrak{s}}_{\text{can}} = \mathfrak{s}_{\text{can}} \otimes K$. More generally, $\bar{\mathfrak{s}}_E = \mathfrak{s}_{\text{can}} \otimes (K \otimes E^{-1})$. Using charge conjugation, the following is therefore easy to see:

Lemma 6.40. *For every $E \in H^2(X; \mathbb{Z})$,*

$$SW_{X,\omega}(E) = \pm SW_{x,\omega}(K - E)$$

6.4.1 Taubes' Results on Symplectic 4-Manifolds and their Consequences

In the years following the advent of SW theory, Taubes proved certain fundamental theorems regarding the SW invariants of symplectic 4-manifolds. One of the most important results is the following:

Theorem 6.41 (Taubes, '95)**.** *Let (X^4, ω) be closed and symplectic with $b_2^+ \geq 2$ and canonical class K. Then*

(i) *$SW_{X,\omega}(0) = \pm SW_{X,\omega}(K)$ equal $\pm 1 \in H_0(\mathcal{B}^*; \mathbb{Z})$.*

(ii) *If $E \in H^2(X; \mathbb{Z})$ is any class with $SW_{X,\omega}(E) \neq 0$, then $0 \leq E \cdot [\omega] \leq K \cdot [\omega]$ with equality in the first relation if and only if $E = 0$, and in the second if and only if $E = K$.*

The proof will be postponed until the next section. Instead, we will study some of the consequences and applications of this theorem. In what follows, X will always be CCOS with $b_2^+ \geq 2$.

Corollary 6.42. *If $SW_X \equiv 0$, then X does not admit a symplectic form.*

Example 6.43.

(i) If X has a Riemannian metric with $s_g > 0$, then X does not admit a symplectic form. This does not hold if $b_2^+(X) = 1$, as exemplified by $\mathbb{C}\mathrm{P}^2$: Equipped with the Fubini-Study metric, it is Kähler-Einstein with positive scalar curvature.

(ii) Let $X_{p,q} = p\mathbb{C}\mathrm{P}^2 \# q\bar{\mathbb{C}\mathrm{P}}^2$. If $p = 1$, these are *blow-ups* of $\mathbb{C}\mathrm{P}^2$, and they are symplectic. If $p, q \geq 2$, exercise 11.4 shows that $SW_{X_{p,q}} \equiv 0$, hence these manifolds are not symplectic. We also have the following result:

Proposition 6.44. *$X_{p,q}$ is almost complex if and only if p is odd.*

Corollary 6.45. *If $p \geq 3$ is odd, then $X_{p,q}$ is almost complex but not symplectic.*

The proposition follows from a more general result:

Lemma 6.46. *A CCOS, simply connected 4-manifold is almost complex if and only if $b_2^+(X)$ is odd.*

Proof. It follows from homotopy theory that a class $k \in H^2(X;\mathbb{Z})$ is the canonical class of an almost complex structure if and only if

$$k^2 = 2\chi(X) + 3\sigma(X) \qquad\qquad k \equiv w_2(X) \mod 2$$

Using the Van der Blij lemma, these conditions imply that $\langle k^2, [X]\rangle \equiv \sigma(X) \mod 8$. Therefore, we have the following equalities modulo 8:

$$0 \equiv k^2 - \sigma(X) = 2(\chi(X) + \sigma(X)) = 4 + 4b_2^+(X)$$

We conclude that $b_2^+(X)$ is odd.

Conversely, suppose that $b_2^+(X) = 2\ell - 1$ is odd: If Q_X is also odd, the Hasse-Minkowski classification tells us that $Q_X = (2\ell-1)(1) \oplus q(-1)$. In the vector space $H^2(X;\mathbb{Z})$, set $k = (3,\ldots,3,1,\ldots,1)$, where there are ℓ entries equal to 3 and $\ell + q - 1$ equal to 1, the last q of which correspond to the negative-definite directions (under the intersection form). Then the reduction of k modulo two equals $(1,\ldots,1)$, which is exactly $w_2(X)$ since $\langle w_2, \alpha\rangle = \langle \alpha, \alpha\rangle \mod 2$. Furthermore,

$$\langle k^2, [X]\rangle = 9\ell + \ell - 1 - q = 5(2\ell - 1) + 4 - q = 2(2 + 2\ell - 1 + q) + 3(2\ell - 1 - q) = 2\chi(X) + 3\sigma(X)$$

where we used that $\chi(X) = 2 + 2b_2(X) = 2 + 2\ell - 1 + q$ and $\sigma(X) = 2\ell - 1 - q$. Hence k is a canonical class.

If Q_X is even, then $Q_X = (2\ell - 1)H \oplus 2mE_8$ by Freedman and Donaldson's theorems. Therefore $2\chi(X) + 3\sigma(X) = 8(\ell + 4|m| - 6m)$. Since $b_2^+(X)$ is odd, we must have at least one copy of H. Choose a basis $\{x, y\}$ for this copy such that $x\cdot x = y \cdot y = 0$ and $x \cdot y = 1$ and set $k = 2x + 2(\ell + 4|m| - 6m)y$. Then k reduces, modulo two, to $0 = w_2(X)$ and $\langle k^2, [X]\rangle = 8(\ell + 4|m| - 6m) = 2\chi(X) + 3\sigma(X)$. $\square$

The result can also be used to prove that certain topological manifolds admit multiple non-diffeomorphic smooth structures. Consider a $K3$ surface, which is Calabi-Yau, hence symplectic. Then $X = K3\#\overline{\mathbb{C}\mathrm{P}^2}$, since the blow-up of a symplectic manifold is symplectic too. In particular, Taubes' theorem ensures that $SW_X \not\equiv 0$. Since $b_2^+(X) = 3$ and $b_2^-(X) = 20$, the intersection form must be $3(1) \oplus 20(-1)$. Freedman's theorem then implies that this manifold is homeomorphic to $X_{3,20} = 3\mathbb{C}\mathrm{P}^2\#20\overline{\mathbb{C}\mathrm{P}^2}$.

However, we showed in exercise 11.4 that $SW_{X_{3,20}} \equiv 0$. Therefore, these manifolds are not oriented diffeomorphic. As a corollary, we see that the existence of a symplectic structure may depend on the choice of smooth structure. That it is also sensitive to the choice of orientation follows from the fact that $\overline{K3}$ does not admit a symplectic form, since $SW_{\overline{K3}} \equiv 0$.

There is another important vanishing theorem for SW invariants:

Theorem 6.47. *If X has a connected sum decomposition $X \cong Y_1\#Y_2$, with $b_2(Y_i) \geq 1$, then $SW_X \equiv 0$.*

An example of this situation is the manifold $(2k+1)\mathbb{C}\mathrm{P}^2$, for $k \geq 1$: it is almost complex, but not symplectic.

Corollary 6.48. *If $X = Y_1 \# Y_2$ has $b_2^+ \geq 2$ and is symplectic, then one of Y_1, Y_2 must have negative definite intersection form.*

We list a few more easy corollaries of Taubes' theorem:

Corollary 6.49. *For X^4 CCOS with $b_2^+(X) \geq 2$, there exist at most finitely many classes $K \in H^2(X;\mathbb{Z})$ which are the canonical class of a symplectic form on X.*

Proof. This follows from the fact that the SW map has finite support. ☐

Corollary 6.50. *X (as above) does not admit a symplectic form whose induced almost complex structure yields a canonical class which is non-zero and torsion.*

Remark 6.51. The case $K = 0$ does occur, for instance for Calabi-Yau manifolds such as $K3$ and T^4.

Proof. Suppose $K \neq 0$ is torsion. By the first part of Taubes' theorem, $SW_{X,\omega}(K) \neq 0$. By the second part, $0 \leq K \cdot [\omega]$ with equality if and only if $K = 0$. Hence $K \cdot [\omega] > 0$, but the intersection with a torsion element must always vanish since the intersection form is bilinear. ☐

Remark 6.52. The necessity of the assumption $b_2^+(X) \geq 2$ is shown by the *Enriques surface* Q, which is the quotient of $K3$ under a free holomorphic $\mathbb{Z}_2$ action. It has fundamental group $\mathbb{Z}_2$, so the cohomology has 2-torsion. It is Kähler (and hence symplectic) with $K \neq 0$ but $2K = 0$: This is possible since $b_2^+(Q) = 1$, as we will now show.

By multiplicative of χ and σ under coverings, $\chi(Q) = \chi(K3)/2$, and $\sigma(Q) = \sigma(K3)/2$. Hence, $\chi(Q) = 12$, $\sigma(Q) = 8$; combining this with $b_1(Q) = 0$ (since $\pi_1(Q) = \mathbb{Z}_2$), we find $b_2^+(Q) = 1$, and $b_2^-(Q) = 9$. As an aside, we note that this is also the simplest example of a manifold which has even intersection form but is not Spin.

Corollary 6.53. *Suppose ω_1 and ω_2 are symplectic forms for X with canonical classes K_1 and K_2.*

(i) *The class $K_2 - K_1$ cannot be torsion and non-zero.*

(ii) *$K_1 = 0 \Rightarrow K_2 = 0$, i.e. if X admits one symplectic form with $K = 0$, then $K = 0$ for any symplectic form.*

Proof. There is a line bundle E such that $\mathfrak{s}_{\mathrm{can},\omega_2} \cong \mathfrak{s}_{\mathrm{can},\omega_1} \otimes E$. By Taubes' theorem, $0 \neq SW_{X,\omega_2}(0) = SW_{X,\omega_1}(E)$. By lemma 2.47, $K_2 = K_1 + 2E$. Now assume $K_2 - K_1 = 2E$ is torsion. Then E is torsion, hence $0 = E \cdot [\omega]$. By Taubes' theorem, this implies $E = 0$, hence $K_2 - K_1 = 0$.

For the second claim, suppose $K_1 = 0$. Since $SW_{X,\omega_1}(E) \neq 0$, $0 \leq E \cdot [\omega] \leq K_1 \cdot [\omega] = 0$. Therefore $E = 0$, hence $K_2 = K_1 = 0$. ☐

Example 6.54. This applies to Calabi-Yau manifolds, showing that any symplectic structure on such a manifold has $K = 0$. We already saw this explicitly for $K3$ and T^4.

Finally, we state another important and difficult theorem by Taubes, which we will not prove:

Theorem 6.55 (Taubes, '96)**.** *Let (X,ω) be a closed symplectic 4-manifold with $b_2^+(X) \geq 2$. Suppose $E \in H^2(X;\mathbb{Z})$ is a class with $SW_{X,\omega}(E) \neq 0$. Then for a generic J compatible with ω, the Poincaré dual class in $H_2(X;\mathbb{Z})$ can be represented by a closed (possibly disconnected) J-holomorphic curve in X.*

Corollary 6.56. *For X as above, if $K \neq 0$, then its Poincaré dual can be represented by a J-holomorphic curve ($SW_{X,\omega}(K) \neq 0$ by Taubes' other theorem). Then, since J-holomorphic submanifolds intersect positively with the fundamental class, $K \cdot [\omega] \geq 0$ with equality if and only if $K = 0$.*

If $0 \neq SW_{X,\omega}(E)$, charge conjugation $SW_{X,\omega}(E) = \pm SW_{X,\omega}(K-E)$ then shows that $E \cdot [\omega] \geq 0$ and $K \cdot [\omega] \geq E \cdot [\omega]$, with equalities if and only if $E = 0$ and $K = E$, respectively: We recover the second statement of Taubes' first theorem.

Remark 6.57. Again, if $b_2^+(X) = 1$, the theorem does not hold and $\mathbb{CP}^2$ gives an easy counterexample: Recall that $K = -3x$ (x the Poincaré dual of the positive generator $[\mathbb{CP}^1] \in H^2(\mathbb{CP}^2;\mathbb{Z})$), hence $K \cdot [\omega] < 0$.

6.4.2 Proof of Taubes' Theorem on SW Invariants of Canonical Classes

In this section, we will outline the proof of theorem 6.41. We start by studying Clifford multiplication no the spinor bundles $V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} = \underline{\mathbb{C}} \oplus K^{-1}$ and $V_- = \Lambda^{0,1}$. For the remainder of this section, let $\Phi_0 = (1,0) \in V_+$ and recall the definition of the Clifford multiplication from section 6.3.3.

Lemma 6.58. *Clifford multiplication by the symplectic form is given by*

$$\gamma(\omega)\Phi_0 = -2i\Phi_0$$

and for $\tau \in \Lambda^{0,2}(X)$, we have

$$\gamma(\tau)\Phi_0 = 2\tau$$

Proof. Exercises 12.2-3. □

Lemma 6.59. *For $\Phi = (\alpha, \beta) \in \Gamma(V_+)$, $\sigma(\Phi,\Phi) \in \Lambda^2_+ \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\,\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}$ is given (component-wise) by*

$$\sigma(\Phi,\Phi)^\omega = \frac{i}{4}\left(|\alpha|^2 - |\beta|^2\right)$$
$$\overline{\sigma(\Phi,\Phi)^{2,0}} = \sigma(\Phi,\Phi)^{0,2} = \frac{1}{2}\bar{\alpha}\beta$$

Proof. We already showed in section 4.3 that

$$\gamma(\sigma(\Phi,\Phi)) = (\Phi \otimes \Phi^\dagger)_0 = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}$$

Restricting the domain, we have $(\Phi \otimes \Phi^\dagger)_0 : V_+ \to V_+;\, \Phi_0 \mapsto (\frac{1}{2}(|\alpha|^2 - |\beta|^2), \bar{\alpha}\beta)$. On the other hand, we can use the previous lemma to see that if ω is the symplectic form, then

$$\frac{i}{4}(|\alpha|^2 - |\beta|^2)\gamma(\omega)\Phi_0 = \left(\frac{1}{2}(|\alpha|^2 - |\beta|^2), 0\right)$$

which shows that $\sigma(\Phi,\Phi)^\omega$ is indeed $\frac{i}{4}(|\alpha|^2 - |\beta|^2)$. Similarly, one sees that $\sigma(\Phi,\Phi)^{0,2}$ must equal $\frac{1}{2}\bar{\alpha}\beta$. □

Lemma 6.60. *There exists a* $\mathrm{U}(1)$*-connection* $\hat{A}_0$ *on* K^{-1} *such that the induced* Spin^c*-connection* A_0 *satisfies* $\nabla^{A_0}\Phi_0 \in \Omega^1(X, K^{-1}) \subset \Omega^1(X, V_+)$.

Proof. Let $\hat{A}$ be any $\mathrm{U}(1)$-connection on K^{-1} and set $\hat{A}_0 = \hat{A} + a$ where $a \in \Omega^1(X; i\mathbb{R})$. Then $\nabla^{A_0}\Phi_0 = \nabla^A\Phi_0 + \frac{1}{2}a\Phi_0$. Define $(\nabla^A\Phi_0)^{0,0} =: b\Phi_0$, where $b \in \Omega^1(X; i\mathbb{R})$, and take $a = -2b$ to obtain the $\mathrm{U}(1)$-connection we want. □

Furthermore, we need to understand what happens to the spinor bundles upon twisting with a line bundle E. Consider $\mathfrak{s}_{\text{can}} \otimes E$, for some $E \in H^2(X;\mathbb{Z})$. Then $V_+ = \underline{E} \oplus (K^{-1} \otimes E)$ and $V_- = \Lambda^{0,1} \otimes E$. Therefore, a positive spinor $\Phi \in \Gamma(V_+)$ is given by $(\alpha, \beta) \in \Omega^{0,0}(E) \oplus \Omega^{0,2}(E)$.

Given a Spin^c-connection A_0 on $V_{+,\text{can}}$ and a $\mathrm{U}(1)$-connection B on E, we obtain a Spin^c-connection $A = A_0 \otimes B$. On the determinant line bundle $L = K^{-1} \otimes E^2$, this yields a $\mathrm{U}(1)$-connection $\hat{A} = \hat{A}_0 \otimes B^2$ with curvature $F_{\hat{A}} = F_{\hat{A}_0} + 2F_B$. Our discussion can be summarized as follows:

Proposition 6.61. *For the* Spin^c*-structure* $\mathfrak{s}_{can} \otimes E$*, the curvature equation for* $(A, \Phi) \in \mathcal{P}$ *with perturbation* η*,*

$$F_{\hat{A}}^{+} = \sigma(\Phi, \Phi) + \eta$$

is equivalent to the following equations for the triple (B, α, β)*:*

$$F_B^{\omega} = \frac{1}{2}\left(\frac{i}{4}\left(|\alpha|^2 - |\beta|^2\right)\omega + \eta^{\omega}\right) - \frac{1}{2}F_{\hat{A}_0}^{\omega}$$
$$F_B^{0,2} = \frac{1}{2}\left(\frac{1}{2}\bar{\alpha}\beta + \eta^{0,2}\right) - \frac{1}{2}F_{\hat{A}_0}^{0,2}$$

On the other hand, we will not prove the following fact:

Proposition 6.62. *The Dirac equation* $D_A^{+}\Phi = 0$ *is equivalent to the equation* $\bar{\partial}_B\alpha + \bar{\partial}_B^{*}\beta = 0$ *for* (B, α, β)*.*

Definition 6.63 (Exterior Covariant Derivative)**.** We define the so-called exterior covariant derivative $\mathrm{d}_B : \Omega^k(E) \to \Omega^{k+1}(E)$ on forms of type $\mu \otimes s \in \Omega^k(X) \oplus \Gamma(E)$ by $\mathrm{d}_B(\mu \otimes s) = (\mathrm{d}\mu) \otimes s + \mu \wedge \nabla^B s$ and linearly extend to define it on $\Omega^k(E)$.

We have the following Weitzenböck-type formula.

Lemma 6.64. *For any section* $s \in \Gamma(E)$*, the following holds:*

$$\bar{\partial}_B^{*}\bar{\partial}_B s = \frac{1}{2}(\mathrm{d}_B^{*}\mathrm{d}_B s - i * (\omega \wedge F_B s))$$

Remark 6.65. Note that $\bar{\partial}^2 \neq 0$ on almost complex manifolds. In fact, for a smooth function f we have $\bar{\partial}^2 f = -N(\partial f)$, where N is the *Nijenhuis tensor*[11]. More generally,

$$\bar{\partial}_B^2\alpha = F_B^{0,2}\alpha - N(\partial_B\alpha)$$

for all $\alpha \in \Gamma(E)$.

We now have everything we need to prove Taubes' first theorem. Since it is already known that $SW_{X,\omega}(E) = \pm SW_{X,\omega}(K - E)$, we just need to prove the following:

(i) $SW_{X,\omega}(0) = \pm 1$.

(ii) If $E \in H^2(X;\mathbb{Z})$ any class with $SW_{X,\omega}(E) \neq 0$, then $0 \leq E \cdot [\omega]$ with equality if and only if $E = 0$.

Consider the SW equations for a specific perturbation η, given by $\eta^{\omega} = F_{\hat{A}_0}^{\omega} - \frac{i}{4}r\omega$ and $\eta^{0,2} = F_{\hat{A}_0}^{0,2}$, where $r \in \mathbb{R}$. The monopole equations then reduce to

$$\bar{\partial}_B\alpha = -\bar{\partial}_B^{*}\beta$$
$$F_B^{\omega} = \frac{i}{8}\left(|\alpha|^2 - |\beta|^2 - r\right)\omega$$
$$F_B^{0,2} = \frac{1}{4}\bar{\alpha}\beta$$

Let $E = 0$, and $r \geq 0$. Then there is an obvious solution: $B = 0$, $|\alpha|^2 = r$, $\beta = 0$, $\bar{\partial}\alpha = 0$. We will show that this solution is unique up to gauge, for sufficiently large r. First, we have to do some calculations.

Let $E \in H^2(X;\mathbb{Z})$ be an arbitrary class. Suppose there exists a solution (B, α, β) to the SW equations. Our last lemma shows that

$$\int_X |\mathrm{d}_B\alpha|^2 \mathrm{vol}_g = \int_X \left(\langle 2\bar{\partial}_B^{*}\partial_B\alpha, \alpha\rangle + \langle i * (\omega \wedge F_B\alpha), \alpha\rangle\right)\mathrm{vol}_g$$

[11] The vanishing of the Nijenhuis tensor is equivalent to integrability of J.

We will now rewrite these two terms. For the first term, we employ the SW equations and the formula for $\bar{\partial}_B^2\alpha$ in terms of the Nijenhuis tensor to find

$$\begin{aligned}\int_X \langle 2\bar{\partial}_B^*\partial_B\alpha, \alpha\rangle \mathrm{vol}_g &= -\int_X \langle 2\bar{\partial}_B^*\bar{\partial}_B^*\beta, \alpha\rangle \mathrm{vol}_g = -\int_X \langle 2\beta, \bar{\partial}_B^2\alpha\rangle \mathrm{vol}_g \\ &= \int_X \Big\langle 2\beta, -F_B^{0,2}\alpha + N(\partial_B\alpha)\Big\rangle \mathrm{vol}_g = \int_X \Big(-\frac{1}{2}|\alpha|^2|\beta|^2 + 2\langle \beta, N(\partial_B\alpha)\rangle\Big)\mathrm{vol}_g\end{aligned}$$

For the second term, we use $\omega \wedge F_B = \omega \wedge F_B^\omega$, the curvature equation and the fact that $\omega \wedge \omega = 2\mathrm{vol}_g$, or equivalently $*(\omega\wedge\omega) = 2$, to find:

$$\begin{aligned}\int_X \langle i*(\omega\wedge F_B\alpha), \alpha\rangle \mathrm{vol}_g &= \int_X \langle i*(\omega\wedge F_B^\omega\alpha), \alpha\rangle \mathrm{vol}_g = -\frac{1}{8}\int_X \langle *(\omega\wedge(|\alpha|^2 - |\beta|^2 - r)\omega\alpha), \alpha\rangle \mathrm{vol}_g \\ &= -\frac{1}{4}\int_X |\alpha|^2\big(|\alpha|^2 - |\beta|^2 - r\big)\mathrm{vol}_g\end{aligned}$$

Putting this all together and isolating the term involving the Nijenhuis tensor, we find:

$$\begin{aligned}2\int_X \langle \beta, N(\partial_B\alpha)\rangle \mathrm{vol}_g &= \int_X \Big(|\mathrm{d}_B\alpha|^2 + \frac{1}{2}|\alpha|^2|\beta|^2 + \frac{1}{4}\big(|\alpha|^2 - |\beta|^2 - r\big)|\alpha|^2\Big)\mathrm{vol}_g \\ &= \int_X \Big(|\mathrm{d}_B\alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{r}{4}(|\alpha|^2 - r)\Big)\mathrm{vol}_g\end{aligned}$$

Consider

$$E\cdot[\omega] = \int_X \frac{i}{2\pi}F_B\wedge\omega = \int_X \frac{i}{2\pi}F_B^\omega\wedge\omega = -\frac{1}{8\pi}\int_X (|\alpha|^2 - |\beta|^2 - r)\mathrm{vol}_g$$

By the so-called Peter-Paul inequality,

$$\int_X 2\langle \beta, N(\partial_B\alpha)\rangle \mathrm{vol}_g \le \int_X \left(\frac{1}{2}|\mathrm{d}_B\alpha|^2 + C|\beta|^2\right)\mathrm{vol}_g$$

for a constant $C = C(g, J, \omega, A_0)$. Adding and subtracting $\frac{1}{4}r|\beta|^2$ to our earlier expression for the term involving the Nijenhuis tensor, we have

$$\begin{aligned}2\int_X \langle \beta, N(\partial_B\alpha)\rangle \mathrm{vol}_g &= \int_X \Big(|\mathrm{d}_B\alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 - 2\pi r(E\cdot[\omega]) + \frac{1}{4}r|\beta|^2\Big)\mathrm{vol}_g \\ &\le \int_X \left(\frac{1}{2}|\mathrm{d}_B\alpha|^2 + C|\beta|^2\right)\mathrm{vol}_g\end{aligned}$$

or, equivalently, we find

$$\frac{1}{2}|\mathrm{d}_B\alpha|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 - 2\pi r(E\cdot[\omega]) + \frac{1}{4}r|\beta|^2 \le C|\beta|^2$$

Now choose $r > 4C$ and generic and suppose $SW_{X,\omega}(0) \neq 0$. Then there must be a solution to the monopole equations for parameters (g, η), hence our above estimate holds. Since all but one of the terms on the left hand side is positive and the $\frac{1}{4}r|\beta|^2$ alone already gives a positive contribution larger than $C|\beta|^2$, we see that necessarily $E\cdot[\omega] \ge 0$.

If $E\cdot[\omega] = 0$, our inequality must be an equation and therefore $\beta = 0$, $|\alpha|^2 = r$ and $\mathrm{d}_B\alpha = 0$. Then α is a non-vanishing section which trivializes E, hence $E = 0$, and B a product connection on $E = X\times\mathbb{C}$. Hence, there is a unique solution (up to gauge) of the SW equations in the case $E = 0$. Since r is generic, we conclude that $SW_{X,\omega}(0) = \pm 1$.